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Quantum phase transitions for two coupled cavities with dipole-interaction atoms

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We investigate the quantum phase transitions for two weakly coupled atom-cavity sites. The interatomic dipole-dipole interaction is considered. Our numerical results show that the dipole-dipole interaction is a crucial parameter for the quantum phase transition. For small atom-cavity detuning, the "superfluid" becomes more and more obvious with the increase of the dipole-dipole interaction. In addition, the strong dipole-dipole interaction can lead the atomic excitation to be suppressed completely, and only the photonic excitation exists for the ground states. When the atom-cavity detuning is comparable with the dipole-dipole interaction, the dipole-dipole interaction enlarges the positive detunings, which is in favor of exhibiting superfluid photonic states. While for the negative detuning, the dipole-dipole interaction will reduce it, and contribute to the formation of the polaritonic insulator states. The cases for extended models have also been briefly analyzed. We also discuss how to find these novel phenomena in future experiments.

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I. INTRODUCTION

The simulation of the strongly correlated many-body systems described by Bose-Hubbard model has received great advances in optical lattices [1–14] and coupled-cavity systems [15–31]. Both of them depend on the competition between the local interaction and the nonlocal tunneling, but there are also some differences between these two basic models. In optical lattices, the quantum phase transition (QPT) of ultracold atoms in periodic potentials is described by the Bose-Hubbard model on site with two atoms interacting and hopping between the adjacent sites. However, in the coupled-cavity systems, two types of particles are usually involved to study the many-body dynamics and its realization relies on the strong light-matter coupling regime. Therefore, the QPT is due to the transferring of the excitations from polaritonic to photonic rather than purely bosonic or purely fermionic entities. However, the realization of the strong coupling in experiment is the greatest bottleneck for the QPT manipulated in the coupled-cavity system. To be optimistic, with the progress in the realization of strong light-matter coupling regime in both atomic [32,33] and solid-state [34,35] cavity quantum electrodynamics devices with single two-level emitters in high-Q resonators, the QPT for coupled-cavity arrays of Jaynes-Cummings (JC) model systems and its variants attract more and more attention.

Subsequent works that deal with coupled nonlinear cavities arrays have addressed the dynamics in the two coupled cavities for its great freedom and flexibility, which can provides a convenience controllable platform for engineering the transport of quantum states via photonic processes and the exact numerical solutions can be easily found. Only very recently the investigation of the dynamical characteristics of such systems has begun. The quantum states transfers [36], the atomic state transfer [37], the Josephson Effect [38], the quantum phase gates [39,40], the entanglement [41,42], the one-excitation dynamics [43], the photon correlations [44], the time evolution of the population imbalance [45], the photonic tunneling effect [46], and the emission characteristics [47] have been studied. However, most of the previous works of this system are limited to the single atom-cavity interactions without consideration of an additional interatomic coupling.

It is well known that the interactions between atoms play an important role in the evolution of the state of the system, especially the dipole-dipole interaction. As the dipole-dipole interaction will be strong at small distances, which can cause the atoms to move or rapidly oscillate around their equilibrium positions, strongly modify the laser excitation of adjacent atoms, and profoundly affect the dynamics in the system. Then lots of novel phenomena appear due to this interatomic coupling [48,49]. The remarkable experimental realization of ultracold dipolar molecules [50] and ultracold atoms with large magnetic moments [51] further promote the research work of probing novel phases of matter that are induced by strong dipole-dipole interaction. For example, Raman transitions between the atoms trapped in different nodes can take place in the two ends of the waveguide based on the effective long-range dipole-dipole interactions between the atoms mediated by the cavity modes [52]. Recently, Alcalde and co-workers [53] found that the critical temperature of the superradiant phase transition could change with the introduction of the dipole-dipole term in taking into account the spatial variation of the coupling coefficient between the pseudospin operators and the boson annihilation and creation operators of the mode excitations. References [54,55] have shown that the dipole interaction between magnetic atoms or polar molecules could stabilize new quantum phases in an optical lattice. To the best of our knowledge, the research of the quantum phase transitions in the system of two coupled cavities with dipole-dipole atomic interaction is absent. Then an investigation of the QPT in this system is also of considerable significance and highly called for. However, it is should be noticed that the terms "insulator" and "superfluid" in this paper are used to indicate the localized and delocalized states, which refer to the states of a small finite system, not true phases in the thermodynamic sense.

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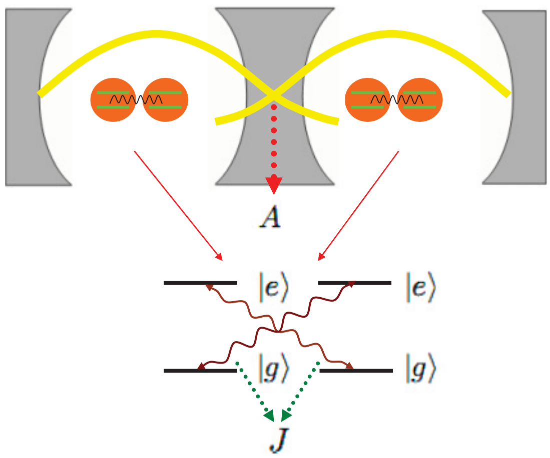

FIG. 1. (Color online) The system consists of two linked identical ultrahigh-finesse single-mode optical cavities, and each cavity contains a pair of dipole-dipole interaction two-level atoms. The two cavities are placed at a certain distance from each other and tunnel coupled by the overlap of the evanescent cavity fields with a hopping strength A. The interaction between the two-level atom and the quantized mode of the cavity field is described by the Jaynes-Cummings model. The direct interaction between the two atoms in each cavity is dipole-dipole type with a strength J.

In the present paper, we develop an optical system to investigate the quantum phase transition in two coupled cavities with dipole-dipole interaction atoms. We find that in the presence of weak cavity-cavity coupling, the atomic dipole-dipole interaction provides an additional controlling parameter, and more importantly, richer physics for the system to appear. The interatomic dipole-dipole interaction will induce new phenomena which are beyond the scope of the previous results.

The paper is organized as follows. We first describe the model under consideration and then simplify it to an effective form, with reformed atomic energy and atom-field coupling strength. We then address the nature of the ground state of the system under a wide range of detuning and dipole coupling strengths. Thirdly, we discuss the quantum phase transitions of the system by choosing three different "order parameters." We finally present our conclusions.

II. TWO COUPLED CAVITIES WITH DIPOLE-DIPOLE INTERACTION ATOMS

The system under consideration is depicted in Fig. 1. It consists of two ultrahigh-finesse single-mode optical cavities, with each cavity containing two coupled identical two-level atoms through dipole-dipole interaction. In experiment we can realize the system by using two Fabry-Perot cavity formed by mirrors (M_1, M_2), into which optically cooled caesium atoms are loaded. Photon hopping can occur by the coupling of the two cavities. Atoms are trapped within the cavities by a far-off resonance trap, which is created by exciting a TEM_{00} cavity mode at $\lambda_F = 935.6$ nm. To achieve a strong coupling we use the $6S_{1/2}$, $F=4 \rightarrow 6P_{3/2}$, $F'=5'$ transition of the $D2$ line in caesium at $\lambda_A = 852.4$ nm, for which the maximum rate of coherent coupling is $g_0/2\pi = 34$ MHz for $(F=4, m_F = \pm 4) \rightarrow (F'=5, m'_F = \pm 5)$. The transverse decay rate for the $6P_{3/2}$ atomic states is $\gamma/2\pi = 2.6$ MHz, and the cavity field decays at rate $\kappa/2\pi = 4.1$ MHz [33]. Besides, the dipole-dipole interaction between the two atoms can be realized in practice by trapping the atoms at a distance equal to the quarter wavelength of a standing-wave laser field. It could be large at small distance, and can be adjusted to zero by a specific position preparation of the atoms or by a specific polarization of the atomic dipole moments. For simplicity we treat the system as an ideal model, neglecting the dissipation induced by atomic spontaneous emission and photons escape from the cavities. The two cavities are weakly coupled by a hopping strength A. Thus we can control the system by regulating the atom-field detuning and the atomic interaction strength. In such a case, the Hamiltonian for the coupled two-cavity system is given by ($\hbar = 1$)

$$
\begin{aligned}
H &= H_1 + H_2 + H_{12},\\
H_1 &= \omega_c a_1^\dagger a_1 + \sum_{i=1,2}[\omega_a \sigma_i^\dagger \sigma_i + g(a_1^\dagger \sigma_i + a_1 \sigma_i^\dagger)]\\
&\quad + J(\sigma_1^\dagger \sigma_2 + \sigma_1 \sigma_2^\dagger),\\
H_2 &= \omega_c' a_2^\dagger a_2 + \sum_{j=3,4}[\omega_a \sigma_j^\dagger \sigma_j + g(a_2^\dagger \sigma_j + a_2 \sigma_j^\dagger)]\\
&\quad + J(\sigma_3^\dagger \sigma_4 + \sigma_3 \sigma_4^\dagger),\\
H_{12} &= A(a_1^\dagger a_2 + a_1 a_2^\dagger),
\end{aligned}
\tag{2.1}
$$

where ω_c, ω_c', and ω_a are the resonance frequencies for cavities and atoms, respectively. Supposing the same atom-field coupling strength in the system, it is defined by an uniform parameter g. $a_i^\dagger$ and a_i are the creation and annihilation operators of the field in cavity i $(i = 1, 2)$. σ_j^+ and σ_j represent the atomic raising and lowering operators of the atom j $(j = 1, 2, 3, 4)$. For static atoms, the coherent dipole-dipole interaction between them can be given by

$$J = |\mathbf{d}|^2 (1 - 3\cos^2\theta)/\mathbf{r}_{12}^3, \tag{2.2}$$

where $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ is the distance between the two atoms located at $\mathbf{r}_1$ and $\mathbf{r}_2$. θ is the angle between $\mathbf{r}_{12}$ and the atomic dipole moment $\mathbf{d}$. Here we assume the dipole moments of the two atoms are parallel to each other and are polarized in the direction perpendicular to the interatomic axis. Then J can be simplified as

$$J = |\mathbf{d}|^2/\mathbf{r}_{12}^3, \tag{2.3}$$

and its strength can be adjusted by changing the positions of the two atoms in each cavity. H_1 and H_2 show the atom-field interaction and the atom-atom coupling in each site, respectively. The cavity-cavity coupling is depicted by H_{12}. Then Hamiltonian H_1 and H_2 can be transformed into two simple forms by a unitary transformation [56],

$$
\begin{aligned}
H_{1\mathrm{eff}} &= U_1^\dagger H_1 U_1 = \omega_c a_1^\dagger a_1 + (\omega_a + J)\sigma_1^\dagger \sigma_1\\
&\quad + \sqrt{2} g (a_1^\dagger \sigma_1 + a_1 \sigma_1^\dagger),\\
H_{2\mathrm{eff}} &= U_2^\dagger H_1 U_2 = \omega_c a_2^\dagger a_2 + (\omega_a + J)\sigma_3^\dagger \sigma_3\\
&\quad + \sqrt{2} g (a_2^\dagger \sigma_3 + a_2 \sigma_3^\dagger).
\end{aligned}
\tag{2.4}
$$

where we assume the frequencies for the two cavities are identical with $\omega_c = \omega_c'$. $U_1 = \exp[-\frac{\pi}{4}(\sigma_1^\dagger\sigma_2 + \sigma_2^\dagger\sigma_1)]$, $U_2 = \exp[-\frac{\pi}{4}(\sigma_3^\dagger\sigma_4 + \sigma_4^\dagger\sigma_3)]$. In the transformed form the dipole coupled atoms are denoted by two fictitious atoms. Only one of them couples to the field mode with frequencies $\omega_a + J$, but the other atom freely evolves decoupling from the field. The effective coupling strength also changes from g to $\sqrt{2}g$. The total excitation for the Hamiltonian H can be defined as $N = N_1 + N_2$ ($N_1 = a_1^\dagger a_1 + \sigma_1^\dagger\sigma_1$ $N_2 = a_2^\dagger a_2 + \sigma_2^\dagger\sigma_2$). We assume the total number of excitations N is conserved and exactly two excitations in the system. In this paper we pay our attention to the strong atom-cavity coupling regime, in which the eigenstates of the individual cavity should be expressed by the dressed states

$$\begin{aligned} |0_i\rangle &= |g_i\rangle|0_i\rangle, \\ |n_i^-\rangle &= \sin\frac{\theta_n}{2}|e_i\rangle|(n-1)_i\rangle - \cos\frac{\theta_n}{2}|g_i\rangle|n_i\rangle, \\ |n_i^+\rangle &= \cos\frac{\theta_n}{2}|e_i\rangle|(n-1)_i\rangle + \sin\frac{\theta_n}{2}|g_i\rangle|n_i\rangle. \end{aligned} \tag{2.5}$$

Where $i = 1, 2$ indicates the cavity number, $\sqrt{n}$ is a photon number state, $\theta_n = \arctan 2\sqrt{2}g\sqrt{n}/(\Delta + J)$, and $\Delta = \omega_a - \omega_c$ is the detuning between the atom and the field. The eigenenergies of these eigenstates are

$$\begin{aligned} E_i^0 &= 0, \\ E_i^{n\mp} &= n\omega_c + \frac{\Delta + J}{2} \mp \frac{1}{2}\sqrt{(\Delta + J)^2 + 8g^2 n}. \end{aligned} \tag{2.6}$$

Using the unitary transformation, the dipole-dipole interaction between atoms is placed in a conspicuous position. It is obvious that both energy levels of the dressed states and excitation probabilities of atoms and field mode are affected by the cooperative action of Δ and J. For $\Delta = 0$, the dressed states only depends on the dipole-dipole interaction, and their energy level difference $E^{n+} - E^{n-} = \sqrt{J^2 + 8g^2 n}$ enlarges with the increase of J. For $\Delta > 0$, Δ and J have consistent effects on the dressed states. While for $\Delta < 0$, the effects of Δ and J cancel each other, and the practical states rely on the larger value of one of them. Therefore the controllable parameter J provides an important and new regime in the dynamical evolution of the coupled-cavity system. The effective form of Hamiltonian H can be rewritten as

$$H_{\text{eff}} = H_{1\text{eff}} + H_{2\text{eff}} + H_{12}. \tag{2.7}$$

Because there are only two excitations in the system, we can write out the possible states of H_{eff} in the order of increasing energy and divide them into five groups, defined as $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$, $|\phi_4\rangle$, $|\phi_5\rangle$, which are corresponding to subspaces $\{|1_1^-\rangle \otimes |1_2^-\rangle\}$, $\{|2_1^-\rangle \otimes |0_2\rangle, |0_1\rangle \otimes |2_2^-\rangle\}$, $\{|1_1^-\rangle \otimes |1_2^+\rangle, |1_1^+\rangle \otimes |1_2^-\rangle\}$, $\{|2_1^+\rangle \otimes |0_2\rangle, |0_1\rangle \otimes |2_2^+\rangle\}$, $\{|1_1^+\rangle \otimes |1_2^+\rangle\}$, respectively. Obviously the energy difference between the adjacent subspaces depends on the parameters in H_{eff}. Then the probability distribution of the ground states in the five subspaces, as well as its nature, will be different for taking different parameter values.

III. THE NATURE OF THE GROUND STATE

We know that strong coupled cavities are in favor of photons hopping between them. However, in practice, the cavity-cavity coupling is usually weak. So without loss of generality, we give out our detailed discussion on the case of $A/g = 0.1$ in the present work. The effects of the cavity-cavity coupling intensity is not the subject we mainly focus on here.

Based on the states in five subspaces, the energy gap ΔE_i ($i = 1, 2, 3, 4$) between $|\phi_j\rangle$ ($j = 1, 2, 3, 4, 5$) can be obtained:

$$\begin{aligned} \Delta E_1 &= \sqrt{(\Delta + J)^2 + 8g^2} \\ &\quad - \tfrac{1}{2}[\sqrt{(\Delta + J)^2 + 16g^2} + (\Delta + J)], \\ \Delta E_2 &= \tfrac{1}{2}[(\Delta + J) + \sqrt{(\Delta + J)^2 + 16g^2}], \\ \Delta E_3 &= \tfrac{1}{2}[\sqrt{(\Delta + J)^2 + 16g^2} - (\Delta + J)], \\ \Delta E_4 &= \sqrt{(\Delta + J)^2 + 8g^2} \\ &\quad - \tfrac{1}{2}[\sqrt{(\Delta + J)^2 + 16g^2} - (\Delta + J)]. \end{aligned} \tag{3.1}$$

These energy gap ΔE_i not only depends on the atom-field coupling intensity g and detuning Δ, but also relies on the atom-atom coupling intensity J. To illustrate the behavior of the ground state, we first pay attention to the resonant condition $\Delta = 0$, and three cases for $J \ll g$, $J \approx g$, and $J \gg g$ are discussed in the following.

When $A, J \ll g$ there is a large energy gap between $|\phi_1\rangle$ and $|\phi_2\rangle$ due to the photon blockade effect, for which the presence of one photon in the cavity blocks the entering of the subsequent photons. In this condition the ground state of the system is approximately $|1_1^-\rangle \otimes |1_2^-\rangle$. Moreover, $|1^-\rangle$ is nearly the maximal entanglement state of atom and field. For this state, only one excitation in each cavity with almost equal probabilities for atomic and field excitations. Thus the ground state is a polaritonic insulator state, which is analogous to the Mott insulator state in the Bose-Hubbard model. It is valid no matter whether $A \leqslant J \ll g$ or $J \leqslant A \ll g$. In Fig. 2 the occupation probabilities of the ground state in the five subspaces for $A = J = 0.1g$ are plotted. The nature of the ground state is reflected from the inset. In fact, the smaller the parameter A is taken, the more obvious this phenomenon appears. For instance, when $A = 0.01g$ even $J = g$, the ground state also shows a similar behavior as the ones depicted in Fig. 2.

Compared to Fig. 2, Fig. 3 indicates a more different behavior for $J \approx g$. Besides $|\phi_1\rangle$, the subspace $|\phi_2\rangle$ is also occupied for the ground state of the system. From its inset we find that both the atom and the field are excited, indicating a polaritonic superfluid state.

When $J \gg g$, the ground state occupies the subspaces $|\phi_1\rangle$ and $|\phi_2\rangle$ with almost the same probabilities, as represented in Fig. 4. However, only the photons are excited in this state. This is because, in this limit, $|n^-\rangle \approx -|gn\rangle$, so the ground state is a delocalized photon state in nature. The state of this form is a photonic superfluid state. Numerical result shows that the ground state is still in photonic superfluid state when $A = 0.01g$ and $J = 10g$.

The results can be understood as follows. At $\Delta = 0$ the energy gap of ΔE_1 is a monotonic decreasing function of J

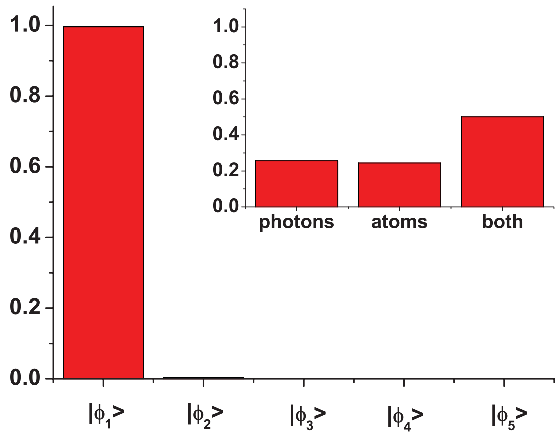

FIG. 2. (Color online) Occupation probabilities of the ground state in the five subspaces. The inset shows the probability distribution of excitations among states with purely photonic, purely atomic, and a mix of them. The hopping strength between the two cavities is weak for $A = 0.1g$, the detuning between the atom and the field is $\Delta = 0$, and the atom-atom coupling strength is $J = 0.1g$. This is a Mott insulator state of polaritons

(see Fig. 5). With the increase of J, the photon blockade effect is destroyed, leading to an increase of the occupied probability of $|\phi_2\rangle$. When $J \approx 10g$, ΔE_1 is almost zero, so the subspaces $|\phi_1\rangle$ and $|\phi_2\rangle$ nearly degenerate. Oppositely, ΔE_2 is a monotonic increasing function of J with a large initial value. Thus the subspace $|\phi_3\rangle$ cannot be occupied. On the other hand, when $A \ll g$, photons hopping between the two cavities are weak. For small value of J, the coupling of atom and field is strong, so that the ground state is in a polaritonic insulator state. With increase of J, the subspace $|\phi_2\rangle$ begins to be occupied, making the system show polaritonic superfluid state. However, large values of J can result in the decoupling of atom and field and thus in favor of photonic excitation. When $J \gg g$, atomic

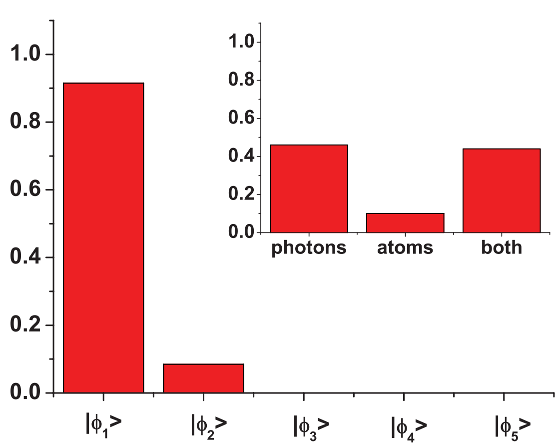

FIG. 3. (Color online) Occupation probabilities of the ground state in the five subspaces. The inset shows the probability distribution of excitations among states with purely photonic, purely atomic, and a mix of them. A and Δ are the same as the ones in Fig. 2. But the dipole-dipole interaction strength is $J = g$. Both the atom and the field are excited, indicating a superfluid state with polaritonic characteristics.

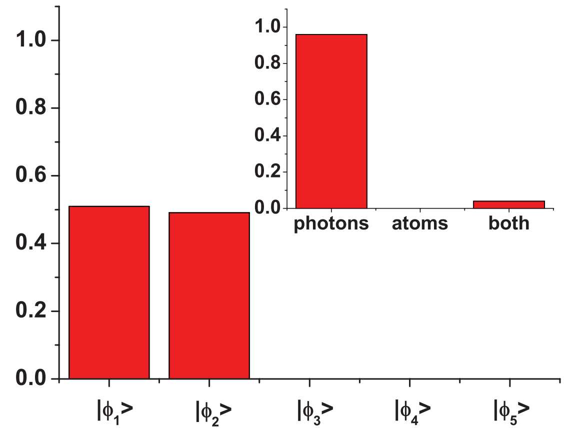

FIG. 4. (Color online) Occupation probabilities of the ground state in the five subspaces. The inset shows the probability distribution of excitations among states with purely photonic, purely atomic, and a mix of them. A and Δ are the same as the ones in Fig. 2. Compared with Figs. 2 and 3, here the dipole-dipole strength is $J \gg g$ with $J = 10g$. This is a superfluid state that is almost entirely photonic in nature.

excitation can be suppressed completely, there is only photonic excitation for the ground states.

Besides, it should be noticed that when the cavity-cavity coupling is pronounced, such as $A = g$, photons hopping is strong. The system is inclined to show a superfluid behavior. When $J \leqslant g$, the ground state appears as a polaritonic superfluid state. While for $J \gg g$, the ground state is in a delocalized photon state, indicating photonic superfluid nature.

Next, we discuss the case of large positive detuning. When $\Delta > 0$, J and Δ have consistent effect on the energy gaps ΔE_i ($i = 1, 2, 3, 4$), which actually depends on $\Delta + J$. Therefore, when $\Delta \gg g$, no matter what value of J is, the energy gap of ΔE_1 is always zero. While ΔE_2 is a monotonic increasing function of $\Delta + J$. Even $J = 0$, the energy gap ΔE_2 is very large. So the ground state has equal occupation probabilities in the subspaces $|\phi_1\rangle$ and $|\phi_2\rangle$, but zero probability in other subspaces, as is shown in Fig. 6. More interestingly, in the large positive detuning limit, the superposition coefficients of

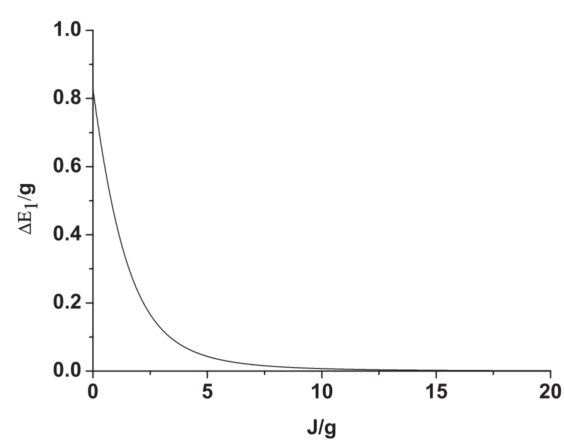

FIG. 5. The energy gap ΔE_1 vs dipole coupling strength J at $\Delta = 0$.

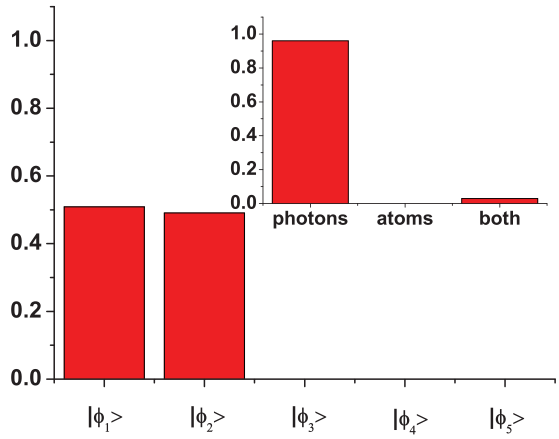

FIG. 6. (Color online) Occupation probabilities of the ground state in the five subspaces. The inset shows the probability distribution of excitations among states with purely photonic, purely atomic, and a mix of them. The hopping strength between the two cavities is weak for $A = 0.1g$, the detuning between the atom and the field is positive, we choose $\Delta = 10g$ and $J = g$. This represents a superfluid state of photons.

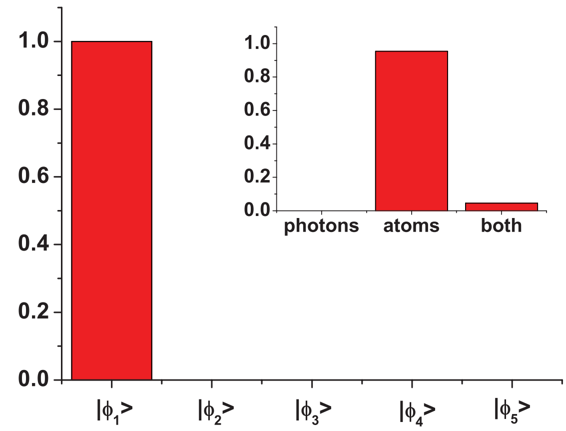

FIG. 7. (Color online) Occupation probabilities of the ground state in the five subspaces. The inset shows the probability distribution of excitations among states with purely photonic, purely atomic, and a mix of them. The hopping strength between the two cavities is weak for $A = 0.1g$, but with large negative atom-field detuning $\Delta = -10g$. Other parameter value is $J = 0.1g$. This is an insulator state composed almost entirely of atomic excitations.

the dressed states in Eq. (2.5) have particular values, $\sin\frac{\theta_n}{2} \approx 0$ and $\cos\frac{\theta_n}{2} \approx 1$. Then, $|n^-\rangle \approx -|g\rangle|n\rangle$ is a delocalized photon state. It can also be confirmed from the inset in Fig. 6, in which the excitation is photonic rather than atomic. As a result, at $\Delta \gg g$, the ground state is in photonic superfluid states. In this case, the effects of A are small. No matter whether $A \ll g$ (e.g., $A = 0.01g$) or larger value of A (e.g., $A = g$), the ground state always indicates photonic superfluid nature.

However, the case for large negative detuning is very different. At $\Delta < 0$ the energy of atoms is smaller than that of photons. In the limit of $-\Delta \gg g, J$ we find $\Delta E_1 \approx |\Delta|$, the ground state is approximately $|1_1^-\rangle \otimes |1_2^-\rangle$. In this condition, $\sin\frac{\theta_n}{2} \approx 1$ and $\cos\frac{\theta_n}{2} \approx 0$. Thus, $|1^-\rangle \approx |e\rangle|0\rangle$. The excitation is almost atomic rather than photonic, standing for an atomic insulator state as is illustrated in Fig. 7. This result can be obtained irrelevant of A. When J approaches the value of $-\Delta$, the effective energy difference between atom and photon is less and less. As a result, the photonic and atomic excitations coexist, as shown in Fig. 8, corresponding to the polaritonic insulator state. With J further increasing to $\Delta + J \geqslant 0$, the system will repeat the results gotten in $\Delta \geqslant 0$, showing polaritonic superfluid nature and then photonic superfluid nature. This tendency will be more evident with the increase of A. For example, when we set $A = g$, $J = -\Delta = 10g$, the system is in a photonic superfluid state.

In this section the constitution and the nature of the ground state for some peculiar parameter values are analyzed. We find that there are four types of states that the ground state may be in. They are the atomic or the polaritonic insulator states, and the polaritonic or the photonic superfluid states. To investigate the nature of the ground state more completely, in the next section we will plot the phase diagrams for a wide range of dipole-dipole interacting intensity J and atom-cavity detuning Δ.

IV. PHASE DIAGRAMS

The phase diagrams of these states can be distinguished using the corresponding "order parameters." In superfluid states the excitations in each cavity are uncertain, resulting in a nonzero variance of the total excitation number ΔN_1 ($\Delta N_1 = \langle N_1^2\rangle - \langle N_1\rangle^2$). Opposite, in the insulator state the number of excitations per cavity is constant and thus has zero variance. However, the insulator state may be either atomic or polaritonic and the superfluid state may be photonic or polaritonic in nature.

To begin with, we should distinguish the insulator and superfluid areas in the phase diagram, which are determined by the variance of the excitation number ΔN_1. In Fig. 9 ΔN_1 is plotted under a wide range values of the atom-cavity detunings Δ and the dipole-dipole interaction strength J. It is apparent that the phase diagram is divided into two sections.

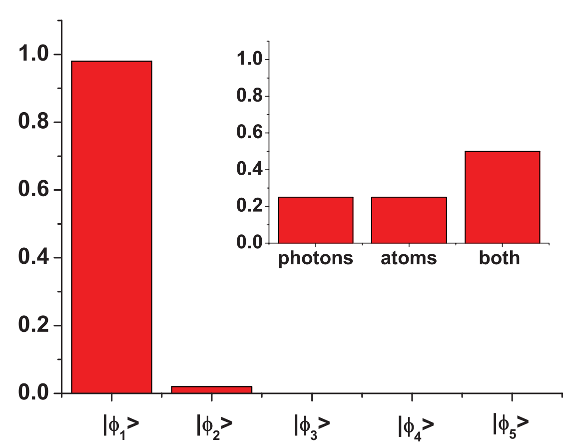

FIG. 8. (Color online) Occupation probabilities of the ground state in the five subspaces. The inset shows the probability distribution of excitations among states with purely photonic, purely atomic, and a mix of them. A and Δ are the same as the ones in Fig. 7, but with $J = 10g$. This is a Mott insulator state of polaritons.

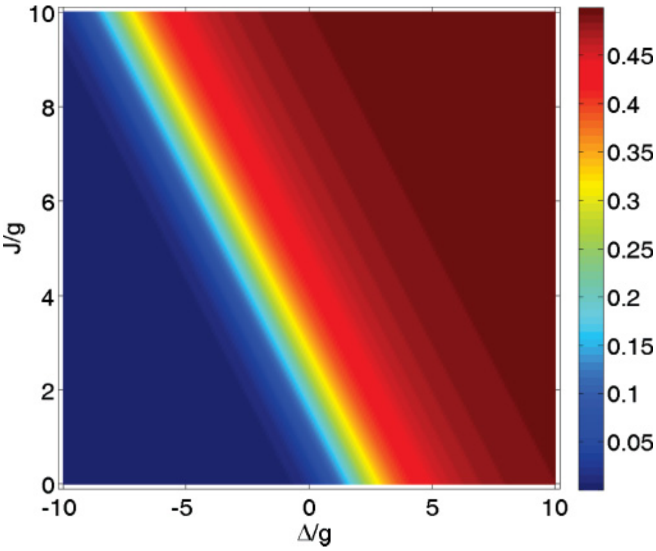

FIG. 9. (Color online) For weak hopping strength $A = 0.1g$, ΔN_1 is plotted vs atom-field detuning Δ and the dipole coupling strength J in the ground state of the coupled two-site and two-excitation system. The phase diagram is mainly divided into two parts: the insulator region (blue area) and the superfluid region (red area)

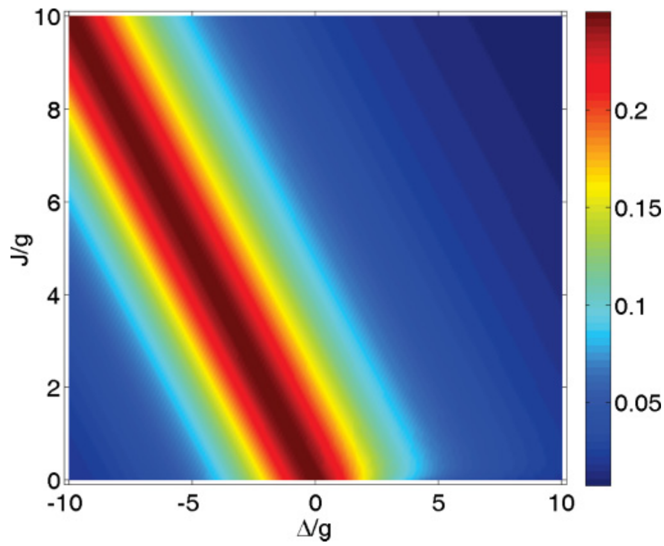

FIG. 10. (Color online) For weak hopping strength $A = 0.1g$, ΔN_{A1} is plotted vs atom-field detuning Δ and the dipole coupling strength J in the ground state of the coupled two-site and two-excitation system. The polaritonic area is presented on the phase diagram (red area), which is mainly located at both sides of the line $J = -\Delta$, and the other area shows the atomic and photonic nature (blue area).

The insulator region is under the boundary where $\Delta < 0$, while above the boundary is the superfluid region. There is also an area which is symmetric to the insulator area, where the ΔN_1 has a maximum value 0.5, indicating a most evident superfluidity.

To determine the allowed types of particles involved in the state, the atomic excitation number variance ΔN_{A1} should be taken as the "order parameter":

$$N_{A1} = \sigma_1^\dagger \sigma_1, \quad \Delta N_{A1} = \langle N_{A1}^2 \rangle - \langle N_{A1} \rangle^2. \tag{4.1}$$

$\Delta N_{A1} = 0$ is corresponding to the atomic insulator state or the photonic superfluid state, while $\Delta N_{A1} > 0$ reveals the polaritonic nature. In Fig. 10 we find that the polaritonic area approximately spreads at both sides of the line $J = -\Delta$. The closer to the line the more obvious the polaritonic nature. In fact, when $J = -\Delta$, $\Delta E_1 = (2\sqrt{2} - 2)g$, it is the conditions for photon blockade effect obviously. Then the ground state only occupies the subspace $|\phi_1\rangle$, with $\sin\frac{\theta_n}{2} = \cos\frac{\theta_n}{2} = \frac{1}{2}$, standing for a maximal entanglement state of the atom and the field. So the ground state indicates polaritonic insulator nature.

So far we may guess that on the basis of what are embodied in Figs. 9 and 10, there must be some overlapping areas in the two figures. In Fig. 11 the product of ΔN_1 and ΔN_{A1} is shown as a contour plot. Only when $\Delta N_1 \Delta N_{A1} > 0$ does it show the polaritonic superfluid characteristic for the ground states of the system. We can clearly identify that there is an area not only in the superfluid region in Fig. 9, but also in the polaritonic area in Fig. 10, representing polaritonic superfluid characters.

Then it is obvious that the phase space in Fig. 9 is divided into four sections, from left bottom to top right corner, in order of atomic insulator region, polaritonic insulator region, polaritonic superfluid region, and photonic superfluid region.

However, the above plotted phase diagrams are just limited to the case of $A = 0.1g$. We know that photons tunneling relies on the coupling between cavities. Large tunneling favors the system showing a superfluid behavior. Reflected in the phase diagrams, large values of A can lead to the enlargement of the superfluid region. So when $A = g$, the boundary of

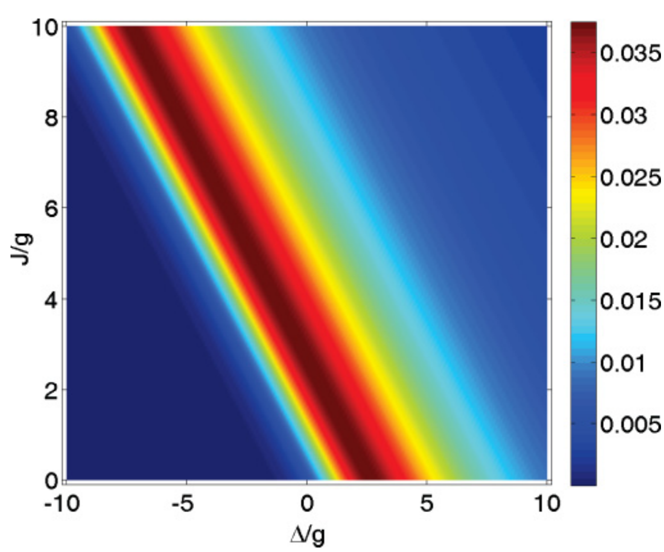

FIG. 11. (Color online) For weak hopping strength $A = 0.1g$, $\Delta N_1 \cdot \Delta N_{A1}$ is plotted vs atom-field detuning Δ and the dipole coupling strength J in the ground state of the coupled two-site and two-excitation system. The polaritonic superfluid area is presented on the phase diagram (red area), which is located above the line $J = -\Delta$. Under the line $J = -\Delta$ is the insulator region both for atom and polariton (blue area). Another area is the photonic superfluid region (blue area), which is located in upper-right corner.

insulator region and superfluid region in Fig. 9 translates to the left with an enlarged superfluid region. While for $A = 0.01$, the result is quite opposite. But the basic behavior of the system showing superfluid-insulator phase transition is still maintained.

V. SUPPLEMENTARY DISCUSSION

In the above sections we consider two identical weakly coupled cavities. The model is simple, but shows the basic behaviors of quantum phase transition. When the two cavities are mismatched with a detuning $\omega_c - \omega_c' = \delta$, photons hopping becomes difficult, making the superfluid more and more weaker. When δ increases to about $0.5g$, in Fig. 12 we find that the distinct superfluid region is confined to a narrow scope, which is close to the previous boundary of insulator region and superfluid region in Fig. 9. The numerical calculation indicates that it corresponds to the polaritonic superfluid state. Below the boundary, the same as in Fig. 9, it refers to the atomic and polaritonic insulator regions. However, due to the detuning between the two cavities, in the light blue area above the boundary, the ground state is in $|0_1^-\rangle \otimes |2_2\rangle$ with almost photonic excitations. In a similar way, when $\delta < 0$, the ground state is in $|2_1^-\rangle \otimes |0_2\rangle$ also with almost photonic excitations. These results suggest that the photons are localized in one of the two cavities.

The present work can also be extended to more than two atoms per cavity, such as N dipole-dipole coupled atoms. The effective Hamiltonian for each cavity can also be written as a fictitious atom interacting with a single-mode field: $H_{\text{eff}} = \omega_c a^\dagger a + [\omega_a + (N-1)J]\sigma^\dagger\sigma + \sqrt{N}g(a^\dagger\sigma + a\sigma^\dagger)$. The collective atom-cavity coupling changes from $\sqrt{2}g$ to $\sqrt{N}g$, and the collective dipole-dipole interaction intensity also increases from J to $(N-1)J$. Then the basic behaviors of the ground state do not change. The nature of the ground state also shows four types of states, and the general forms of the phase diagrams are similar to the ones of the two atoms cases. The difference is the values of the controlling parameters will change accordingly.

We also can replace the two two-level atoms by two three-level atoms, such as two Λ configuration three-level atoms (with a excited state $|3\rangle$ and two ground states $|1\rangle$ and $|2\rangle$) in each cavity. Then there are more ways to realize the dipole-dipole interaction. For example, there are two ways for the resonant dipole-dipole interaction [57]. The transitions $|1\rangle \leftrightarrow |3\rangle$ of the first atom and the same transitions of the second atom are coupled by dipole-dipole interaction. Similar dipole-dipole interaction also exists in the transitions $|2\rangle \leftrightarrow |3\rangle$ of the first atom and the same transitions of the second atom. Therefore the system may exhibit richer behaviors as there are more controllable parameters.

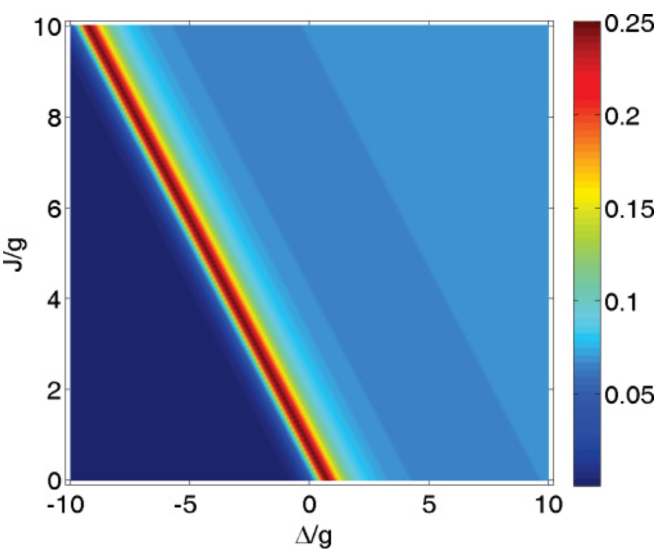

FIG. 12. (Color online) We assume that the two cavities are weakly coupled with $A = 0.1$ and not identical with a detuning $\delta = 0.5$. ΔN_1 is plotted vs atom-field detuning Δ and the dipole coupling strength J in the ground state of the coupled two-site and two-excitation system. The phase diagram is mainly divided into three parts: the insulator region (dark blue area), the superfluid region (red area), and the region where photons are localized in one of the cavity (light blue area).

VI. CONCLUSIONS

In summary, we have investigated the QPT of a system composed of two weakly coupled cavities, each containing a pair of two-level atoms with dipole-dipole interaction. In the conditions of fixed cavity-cavity interaction and atom-cavity coupling strength, the nature of the ground state is dependent on the constituents of the dressed states in each cavity and the occupation probabilities of the ground state in the five subspaces. Moreover, both of them attribute to the dipole-dipole interaction strength between the localized atoms and the atom-field detuning in each cavity. By choosing three different order parameters, we found that the ground state of the system represented richer behaviors than the Bose-Hubbard model. Four types of states are revealed, which divide the phase space into four regions. They are the atomic insulator state, the polaritonic insulator state, the polaritonic superfluid state, and the photonic superfluid state. In the scope of parameter values we have taken in this paper, the insulator or superfluid phases is determined by the combinative effect of Δ and J, that is the value of $\Delta + J$. Small negative values of it is in favor of polaritonic insulator states, while for small positive values of it embodies polaritonic superfluid state. The larger the negative value of $\Delta + J$ is taken, the more obvious the atomic insulator nature appears, and oppositely it shows photonic superfluid nature. Our work can also be extended to the models of two mismatched cavities or more than two atoms in each cavity as well as three-level cases. The QPT for the latter two extended models may become richer because of more controllable parameters.

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