Necessary and sufficient conditions for local creation of quantum correlation

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Quantum correlation can be created by a local operation from some initially classical states. We prove that the necessary and sufficient condition for a local channel to create quantum correlation is that it is not a commutativity-preserving channel. This condition is valid for arbitrary finite dimension systems. We also derive the explicit form of commutativity-preserving channels. For a qubit, a commutativity-preserving channel is either a completely decohering channel or a mixing channel. For a three-dimensional system (qutrit), a commutativity-preserving channel is either a completely decohering channel or an isotropic channel.

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In this article we derive a simple necessary and sufficient condition for a local channel to create quantum correlation in some half-classical states, which is valid for arbitrary finite dimension systems. A trace-preserving local channel can create quantum correlation if and only if it is not a commutativity-preserving channel. For the qubit case we show that a commutativity-preserving channel is either a mixing channel or a completely decohering channel. This confirms the result in Ref. [24]. For the qutrit case, quantum correlation can be created by a local channel in some half-classical input states if and only if the channel is neither a completely decohering channel nor an isotropic channel. We also analyze the reason for a local mixing channel to create quantum correlation in a qutrit situation and then give a conjecture to extend the result of qutrits to arbitrary finite-dimension systems.

The total correlation between two quantum systems is composed of classical and quantum correlations. From this point of view, quantum correlation is defined as the difference between total and classical correlations. Therefore, various measures of quantum correlation defined for one party of a composite system vanish for exactly the same class of states, called half-classical states. Because classical correlation is defined by the correlation that can be revealed by local measurements, a state $\rho_{AB}$ is half classical on $B$ if and only if there exists a measurement on $B$ that does not affect the total state. As proved in Ref. [25] a half-classical state on $B$ can be written as

$$\rho_{AB} = \sum_i p_i \rho_A^{\alpha_i} \otimes |\alpha_i\rangle_B \langle \alpha_i|,$$

where $|\alpha_i\rangle_B$ consist of an orthogonal basis for the Hilbert space of subsystem $B$, and $\rho_A^{\alpha_i}$ are corresponding density matrices of $A$. The subsystem $A$ can be a single quantum particle or an ensemble of quantum particles. In the following, by quantum correlation, we mean quantum correlation defined on subsystem $B$. The main purpose of this paper is to characterize the channel $\Lambda_B$ satisfying

$$I_A \otimes \Lambda_B(\rho_{AB}) \in D_0 \quad \forall \rho_{AB} \in D_0,$$

where $D_0$ is the set of half-classical states. Before providing the condition, we first introduce a class of quantum channels, which we call commutativity-preserving channels.

**Definition 1 (commutativity-preserving channel).** A commutativity-preserving channel $\Lambda_{CP}$ is the channel that can...
preserve the commutativity of any input density operators; i.e.,
\[ [\Lambda^{CP}(\xi), \Lambda^{CP}(\xi')] = 0 \] (3)
holds for any density operators satisfying \([\xi, \xi'] = 0\).

It is worth mentioning an equivalent definition of a commutativity-preserving channel. A channel \(\Lambda\) is a commutativity-preserving channel if and only if
\[ [\Lambda(\phi), \Lambda(\psi)] = 0 \] (4)
holds for any pure states satisfying \((\phi|\psi) = 0\). The “only if” part is obtained directly by choosing \(\xi = |\phi\rangle\langle\phi|\) and \(\xi' = |\psi\rangle\langle\psi|\). Conversely, if Eq. (4) holds, by writing \(\xi\) and \(\xi'\) on their common eigenbasis, we arrive at Eq. (3).

Now we are ready to prove the first main result of this paper. It holds for arbitrary finite-dimensional systems.

\textbf{Theorem 1.} A channel \(\Lambda\) acting on subsystem \(B\) can create quantum correlation between subsystems \(A\) and \(B\) for some input half-classical state \(\rho_{AB}\) if and only if it is not a commutativity-preserving channel.

**Proof.** Any separable state can be written as
\[ \xi_{AB} = \sum_i p_i \xi_i^A \otimes \xi_i^B, \] (5)
where \(\xi_i^A\) are linearly independent. We will first prove that \(\xi_{AB}\) is a half-classical state if and only if
\[ [\xi_i^A, \xi_j^A] = 0 \quad \forall \ i, j. \] (6)
For proving the “only if” part, we notice that for any half-classical state, there exists a measurement basis \(\Pi_B\) that does not affect the state. Therefore,
\[ \sum_i p_i \xi_i^A \otimes (\xi_i^B - \Pi_B \xi_i^B \Pi_B) = 0. \] (7)
Because \(\xi_i^A\) are linearly independent, \(\xi_i^B\) is diagonal on \(\{\Pi_B\}\) and thus satisfies Eq. (6). Conversely, if Eq. (6) holds, \(\xi_i^B\) and \(\xi_j^B\) have eigenvectors in common for any \(i\) and \(j\). By choosing these eigenvectors as the basis for von Neumann measurement, the state does not change after the measurement, which means that \(\xi_{AB}\) is a half-classical state. Now consider an arbitrary half-classical state in the form of Eqs. (5) and (6) as the input state, the channel \(\Lambda\) acting on subsystem \(B\) leads the state to \(\xi_{AB}' = \Lambda_A \otimes \Lambda_B(\xi_{AB}) = \sum_i p_i \xi_i^A \otimes (\Lambda(\xi_i^B))\), which is still a half-classical state if and only if
\[ [\Lambda(\xi_i^B), \Lambda(\xi_j^B)] = 0, \] (8)
for arbitrary choice of \(\xi_i^B\) and \(\xi_j^B\) satisfying Eq. (6). This is just the definition of a commutativity-preserving channel. Therefore, the channel \(\Lambda\) can create quantum correlation for some input half-classical states if and only if it is not a commutativity-preserving channel. This completes the proof.

The rest of this paper is devoted to exposing the exact form of a commutativity-preserving channel.

Since \([1, \rho] = 0, \forall \rho\), we obtain a necessary condition for a commutativity-preserving channel
\[ [\Lambda(I), \Lambda(\rho)] = 0 \quad \forall \rho. \] (9)
When \(B\) is a qubit, Eq. (9) is also the sufficient condition. The reason is as follows. By using the linearity of \(\Lambda\), the left-hand side of Eq. (4) can be written as
\[ \frac{1}{2} \left[ \Lambda(\langle\phi|\phi\rangle + |\psi\rangle\langle\psi|), \Lambda(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|) \right] \]
\[ = \frac{1}{2} \left[ \Lambda(I), \Lambda(u\sigma^z u^\dagger) \right], \] (10)
where \(|\psi\rangle = u|0\rangle\) and \(|\phi\rangle = u|1\rangle\). Since any qubit state \(\rho\) can be decomposed as \(\rho = \frac{1}{d+1} \sum_{i=0}^{d} \lambda_i \Pi_i\), Eq. (4) is equivalent to Eq. (9). From this observation, we can see that a qubit channel \(\Lambda\) is commutativity-preserving if and only if it is one of the following two cases:

Case 1. \(\Lambda(I) = I\), which means that \(\Lambda\) is a unital channel. Here we define a mixing channel \(\Lambda^M\) as
\[ S(\Lambda^M(\rho)) \geq S(\rho) \quad \forall \rho, \] (11)
where \(S(\rho) \equiv -\text{Tr}(\rho \log_2 \rho)\) is the von Neumann entropy. It is worth mentioning that when a channel is a mixing channel, its extension to larger systems \(I_A \otimes \Lambda^M\) is still a mixing channel.

As proved in Ref. [26], a mixing channel is equivalent to a unital channel.

Case 2. \(\Lambda(I) \neq I\). Then the diagonal basis of \(\Lambda(I)\) is specified. According to Eq. (8), the two matrices \(\Lambda(\rho)\) and \(\Lambda(I)\) share eigenvectors. In other words, the channel \(\Lambda\) takes any input state \(\rho\) to a diagonal form on the eigenbasis of \(\Lambda(I)\) and is thus a completely decohering channel.

Therefore, when \(B\) is a qubit, a commutativity-preserving channel is either a mixing channel or a completely decohering channel. This confirms the result in Ref. [24].

In the following, we will move on to study the exact form of a commutativity-preserving channel for high-dimension cases.

\textbf{Definition 2 (isotropic channel).} An isotropic channel is of the form
\[ \Lambda^{iso}(\rho) = p \Gamma(\rho) + (1 - p) I, \] (12)
where \(\Gamma\) is any linear channel that preserves the eigenvalues of \(\rho\). According to Ref. [27], \(\Gamma\) is either a unitary operation or unitarily equivalent to transpose. Parameter \(p\) is chosen to make sure that \(\Lambda\) is a completely positive channel. In particular, \(-1/(d-1) \leq p < 1\) when \(\Gamma\) is a unitary operation, and \(-1/(d-1) < p \leq 1/(d+1)\) when \(\Gamma\) is unitarily equivalent to transpose.

\textbf{Theorem 2.} Consider the half-classical input state in Eq. (1) with \(B\) a qutrit, a channel \(\Lambda\) cannot create a quantum correlation in any half-classical input state if and only if \(\Lambda\) is either a completely decaying channel or an isotropic channel.

**Proof.** Write the eigendecomposition of \(\Lambda(I)\) as
\[ \Lambda(I) = \sum_{i=1}^{N} \lambda_i I_{r_i}. \] (13)
Here \(\sum_{i=1}^{N} r_i = \sum_{i=1}^{N} \lambda_i = 3, \lambda_i \geq 0, r_i\) are positive integers, and \(I_{r_i}\) are identities of the \(r_i\)-dimensional subspace \(V_{r_i}\). From Eq. (9) we have
\[ \Lambda(\rho) = \sum_{i=1}^{N} q_i \xi_i^B \quad \forall \rho, \] (14)
where \(\xi_i^B\) is a density operator on \(V_{r_i}\).

Clearly, when the eigenvectors of \(\Lambda(I)\) are nondegenerate, i.e., \(N = 3\) and Eq. (13) becomes \(\Lambda(I) = \sum_{i=1}^{3} \lambda_i I_{r_i}\), the channel \(\Lambda\) is a completely decohering channel, since it takes
any input state $\rho$ to a diagonal form on basis $\{\Pi_i\}$. When two or three eigenvectors of $\Lambda(\rho)$ are degenerate, we study the eigendecomposition of $\Lambda(\phi)$ for a pure input state $|\phi\rangle$:

$$\Lambda(\phi) = \sum_{i=1}^{N^\phi} \lambda_i^2 I_{r_i(\phi)},$$

where $N^\phi \geq N$ and $V_{r_i(\phi)} \subseteq V_{r_i}$. When none of $\Lambda(\phi)$ break the degeneracy of eigenvectors of $\Lambda(I)$, i.e., $N^\phi = N$ and $V_{r_i(\phi)} = V_{r_i}$, the channel is also a completely decohering channel. Now we focus on the case that some $\Lambda(\phi)$ can break the degeneracy of eigenvectors of $\Lambda(I)$, i.e., $N^\phi > N$ and $V_{r_i(\phi)} \subset V_{r_i}$ for some $i$. Let $\{|\phi_k\rangle\}_{k=0}^2$ be a basis of the three-dimensional Hilbert space and $|\phi_k\rangle$ be the pure input state whose corresponding output state $\Lambda(\phi_k)$ has the most different eigenvalues. It means that $N^\phi_k > N^\phi$ for $\phi_k$.

**Case 1.** For any state $|\phi_0\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle$ that is orthogonal to $|\phi_0\rangle$, we have $N^\phi_0 = N^\phi_k$ and $V_{r_i(\phi_0)} = V_{r_i(\phi_k)}$. Then for an arbitrary input state $\phi$ such that $\phi_k = \sum c_i^2 |\phi_i\rangle$, $\Lambda(\phi)$ has the following form:

$$\Lambda(\phi) \Lambda(c_i(\phi_i|^\phi_i\rangle - c_j|^\phi_j\rangle)) = 0.$$ (16)

Therefore, $\Lambda(\phi)$ is diagonal on the same basis as $\Lambda(\phi_0)$.

**Case 2.** There exists a pure state, e.g., $|\phi_2\rangle$, whose corresponding output state $\Lambda(\phi_2)$ does not break as much degeneracy as $\Lambda(\phi_0)$, i.e., $N^\phi_2 < N^\phi_0$. We will first prove that for any pure state $|\phi_0\rangle = |\phi_0\rangle - |\phi_0\rangle + |\phi_0\rangle$ in two-dimensional subspace $V_{r_0}$, the output state is diagonal on the same basis as $\Lambda(\phi_0)$, e.g., $\{\Pi_i\}$. We introduce $|\psi(\beta)\rangle = |\phi_1\rangle + \beta|\phi_2\rangle$ and $|\psi(\phi_0)\rangle = \beta^*|\phi_0\rangle - |\phi_2\rangle$. Notice that $\langle \psi(\phi_0)\beta|\phi(\beta)\rangle = 0$, and we have

$$\Lambda(\phi_0)(\beta), \Lambda(\phi(\beta)) = 0.$$ (17)

Because $\sum_{k=0}^2 |\phi_k\rangle = 1$, we have $N^\phi_0 = N^\phi_k$ and $V_{r_i(\phi_0)} = V_{r_i(\phi_k)}$. Therefore, $\Lambda(\phi(\beta))$ is diagonal on $\{\Pi_i\}$ by noticing that $\Lambda(\phi_0(\beta)), \Lambda(\phi(\beta)) = 0$. Since the channel cannot increase the distance between states, $\Lambda(\phi(\beta))$ breaks the same degeneracy as $\Lambda(\phi_0)$ for sufficiently large $|\beta|$. From Eq. (17), we have $\Lambda(\phi(\beta))$ and $\Lambda(\phi(\beta))$ are diagonal on $\{\Pi_i\}$. Therefore, $\Lambda(\phi(\beta)) = \Lambda(\phi(\beta)) + \Lambda(\phi(\beta)) - |\beta|^2 \Lambda(\phi(\beta))$ is also diagonal on $\{\Pi_i\}$. Further, we will show that $\Lambda(\phi(\beta))$ is diagonal on $\{\Pi_i\}$ for arbitrary $\beta$. From Eq. (17), this is obvious when $\Lambda(\phi_0(\beta))$ is nondegenerate. For the case where $\Lambda(\phi_0(\beta))$ is degenerate, $\Lambda(\phi_0(\beta)) = |\beta|^2 \Lambda(\phi_0) + \Lambda(\phi(\beta))$ is also diagonal on $\{\Pi_i\}$. Therefore, $\Lambda(\phi(\beta))$ is diagonal on $\{\Pi_i\}$. A suitable change of basis in $\Lambda(\phi(\beta))$ is diagonal on $\{\Pi_i\}$. A is a completely decohering channel.

**Case 3.** Now we are left only with the case that $N^\phi_2 = N^\phi_0$ but $I_{r_i(\phi_2)} \neq I_{r_i(\phi_0)}$, which can happen only when $\Lambda(I) = I$ and $N^\phi = 2$. Therefore, we have

$$\Lambda(\phi_k) = p\Pi(\phi_k) + (1 - p)\frac{I}{3},$$

where $\Pi(\phi_k)$ is a basis determined by $|\phi_k\rangle$. Notice that $p$ is independent of $\phi_k$ because of the linearity of $\Lambda$. Consequently, for any input state $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$, we have $\Lambda(\rho) = p \sum_i p_i \Pi(\phi_i) + (1 - p)I/3$. It means that channel $\Lambda$ is an isotropic channel.

Combining the three cases, we conclude that for a qudit, a commutativity-preserving channel is either a completely decohering channel or an isotropic channel.

Since the depolarizing channel is a subset of mixing channel, there exist mixing channels that are able to locally create quantum correlation. Therefore, mixedness can contribute to creation of quantum correlations. Here we give an example to look more closely at why a mixing channel can create quantum correlation in states with high dimensions. Consider the mixing channel $\Lambda(\cdot) = \sum_i E^{(i)}(\cdot)E^{(i)},$ where the Kraus operators are

$$E^{(i)} = |2\rangle\langle 2|,$$

$$E^{(i)} = e_i u^{(i)}_2 |0\rangle\langle 0| + |1\rangle\langle 1| = 1, 2, \ldots.$$ (19)

Here $u^{(i)}_2$ are rank-2 unitary operators on basis $\{|0\rangle, |1\rangle\}$. This channel can create quantum correlation in the state $\rho = \tilde{\rho}_A \otimes |\psi\rangle\langle\psi|$. Therefore, average teleportation fidelity $F$ is related to the maximum singlet fraction (MSF) $F = \max_\Phi \langle \Phi|\rho|\Phi\rangle$ as $f = (dF + 1)/(d + 1)$. After the action of a mixing channel on $B$, the MSF becomes

$$F' = \text{Tr}(\rho \Xi),$$

where $\Xi = \sum_i |i\rangle\langle 0| \otimes E^{(i)}(\cdot)E^{(i)}$. Notice that for a mixing channel $\Lambda(\cdot) = \sum_i E^{(i)}(\cdot)E^{(i)}$, its conjecture $\Lambda^*(\cdot) = \sum_i E^{(i)}(\cdot)E^{(i)}$ is also a mixing channel. Therefore, $\Xi_A = \Xi_B = \Xi/2$, so $\Xi$ can be decomposed as a mixture of maximally entangled pure states $\Xi = \sum_i p_i |\Phi_i\rangle\langle\Phi_i|$. Then we have $F' = \sum_i p_i \langle \Phi_i|\rho|\Phi_i\rangle \leq F$. Therefore, average teleportation fidelity can never be increased by a mixing channel. This result suggests that quantum correlation created by a mixing channel may not be a useful resource for quantum-information tasks.

In summary, we have proved that the necessary and sufficient condition for a local operation to create quantum correlation in some half-classical state is that it is not a commutativity-preserving channel. When the subsystem $B$ affected by the
local channel is a qubit, a commutativity-preserving channel is either a mixing channel or a completely decohering channel. This result confirms the results in Ref. [24]. When $B$ is a qutrit, we have proved that a commutativity-preserving channel is either an isotropic channel or a completely decohering channel. This result is likely to be extended to arbitrary finite dimension situation.

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