Feynman relation of Bose-Einstein condensates with spin-orbit coupling

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We find that the Feynman relation of Bose-Einstein condensates with spin-orbit coupling, which relates the energy of excitations, the static structure factor in condensed phase, and the dispersion of free bosons, is not satisfied in the whole momentum space. The dispersion is highly anisotropic and more divergent in the infrared limit compared to that without spin-orbit coupling because of spontaneous breaking of the O(2) symmetry of the ground state. And the dispersion also exhibits time reversal asymmetry for plane-wave condensates, which is condensed on a momentum with a finite value. We also find that larger spin-orbit coupling makes the excitations out of condensates more coherent.

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I. INTRODUCTION

Spin-orbit coupling (SOC) plays a crucial role in a variety of physical systems ranging from atoms and nuclei to topological insulator [1] and spintronics [2]. By controlling atom-light interaction, one can generate an artificial external nonabelian gauge potential coupled to neutral atoms [3–7] which takes the form of Rashba-Dresselhaus SOCs, familiar in semiconductor physics [8,9]. Recently, a system of spin-orbit-coupled ultracold bosons was realized by Lin *et al.* [10]. New quantum states may be anticipated in these systems [11–22]. Interesting density patterns have been observed in the theoretical simulations for the condensates, with Refs. [18–20] and without Refs. [21–24] rotation.

With the presence of two-dimensional isotropic Rashba SOC, the ground state of free bosons is a ring with $|\mathbf{k}| = \lambda$ in momentum space [13,14], which has infinite degeneracy, where λ is the strength of the SOC. Taking into account s-wave scattering processes between bosons, the full Hamiltonian of the system possesses O(2) symmetry in momentum space. Mean-field analysis [13] shows that the interactions between bosons break the O(2) symmetry of the ground state and lift the infinite degeneracy. In this approximation, it is found that bosons prefer to condense at single momentum on the ring if the interactions between particles satisfy $g_{\uparrow\uparrow}g_{\downarrow\downarrow} > g_{\uparrow\downarrow}^2$, where $g_{\uparrow\uparrow}$ and $g_{\downarrow\downarrow}$ are the strengths of interactions between the same components of the bosons, and $g_{\uparrow\downarrow}$ is that between different components of the bosons. In this case, the condensate has plane-wave (PW) order. When $g_{\uparrow\uparrow}g_{\downarrow\downarrow} < g_{\uparrow\downarrow}^2$, bosons prefer to condense at two points with opposite momenta on the ring. This gives the striped phase (SP). The properties of the excitations over the condensates have been discussed intensely. These include the dispersion relations [18,25–28], superfluid critical velocity [25], mapping of the system to the chiral magnetism [18], effective scattering vertex using renormalization-group analysis [15], and effect of excitations on the stability of condensates [26,27].

However, the relation between the energy of the fluctuations and the corresponding correlation functions has not been investigated. In bosonic condensed systems without SOC, there is a simple *Feynman relation* between the energy of the excitations $\omega_{\mathbf{k}}$ and the static structure factor $S(\mathbf{k})$ at the same momentum [29]:

$$\omega_{\mathbf{k}} = \frac{\varepsilon_0(\mathbf{k})}{S(\mathbf{k})}, \tag{1}$$

where $\varepsilon_0(\mathbf{k})$ is the dispersion for free bosons. In this paper, we investigate whether the Feynman relation still holds when SOC is introduced. We find that the excitations over condensates consist of two bands. And due to breaking of the O(2) symmetry of the ground state, the dispersion is highly anisotropic and more divergent in the infrared limit than that without SOC. The dispersion also breaks time reversal symmetry because the bosons are condensed onto a single momentum with a finite value. SOC enhances the coherence of bosons and the effect is more significant when its strength is larger. The Feynman relation is not applicable in the whole momentum space.

This paper is organized as following: We map the Lagrangian of the system to a nonlinear σ model which has been modified by SOC in Sec. I. In Secs. II and III, the dispersion relation for the excitations and the static structure factor are obtained, respectively. Based on these results, we find that the Feynman relation is not applied in the whole momentum space. Part IV is a summary of this work.

II. MODIFIED NONLINEAR σ model

A. Model

The model of a two-dimensional homogeneous boson system with the presence of SOC is written in Lagrangian form as

$$Z = \int [d\Psi^*, d\Psi] e^{-L},$$
$$L = \int_0^\beta d\tau d\mathbf{r} \Psi^*(\mathbf{r},\tau)\left(\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 + 2i\lambda\nabla\cdot\sigma - \mu\right)\Psi(\mathbf{r},\tau)$$
$$+ \frac{1}{2} g \int_0^\beta d\tau d^2\mathbf{r} n(\mathbf{r},\tau)^2, \tag{2}$$

where $\Psi(\mathbf{r},\tau) = (\psi_\uparrow(\mathbf{r},\tau), \psi_\downarrow(\mathbf{r},\tau))$. $\Psi(\mathbf{r},\tau)$ are boson fields. They are complex numbers in a path integral scheme due to the bosonic properties of the particles. $\Psi^*(\mathbf{r},\tau)$ are complex

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conjugates of fields $\Psi(\mathbf{r},\tau)$. $n(\mathbf{r},\tau)=\Psi^*(\mathbf{r},\tau)\Psi(\mathbf{r},\tau)$ is the density of bosons. $\mathbf{r}$ and τ are the space and imaginary time coordinates. $\beta=\frac{1}{k_BT}$, where k_B is the Boltzmann constant and T is the temperature of the system. In this paper, we consider the zero-temperature case. The spin indexes $\uparrow,\downarrow$ are pseudospins, which usually refer to degenerate bands, such as in [30]. λ is the strength of SOC. It couples the psedudospins with the movement directions of bosons. μ is the chemical potential of the system. m is the mass of bosons. $g=4\pi a\hbar^2/m$ is the positive interacting coupling constant fixed by the s-wave scattering length a. In this paper, we consider the symmetric point for particle-particle interactions; that is, the interaction strength between different spins are the same as that between the same spins. In typical experimental situations, the strengths of spin-dependent interactions are always much lower than those for spin-independent terms [10]. We expect that the result derived with symmetric interactions should also be applicable to cases where the interactions are not symmetric but PW order is still preferred. We set $\hbar=2m=1$ for simplicity.

B. Modified nonlinear σ model

In boson systems without SOC, the method of the nonlinear σ model has been proved successful [31]. With SOC, we write the bosonic field $\Psi(\mathbf{r},\tau)$ into its amplitude $\nu(\mathbf{r},\tau)=\sqrt{n(\mathbf{r},\tau)}$ and phase $\mathbf{z}(r,\tau)$,

$$\Psi(\mathbf{r},\tau)=\nu(\mathbf{r},\tau)\mathbf{z}(\mathbf{r},\tau), \tag{3}$$

where $\mathbf{z}(\mathbf{r},\tau)$ are complex two-component vectors under the constraint $|\mathbf{z}|^2=1$. In terms of ν and $\mathbf{z}$, the Lagrangian in Eq. (2) can be written as

$$\begin{aligned}L=\int d\tau d\mathbf{r}\Big[&\nu^2\mathbf{z}^*\partial_\tau\mathbf{z}+(\nabla\nu)^2+\nu^2\nabla\mathbf{z}^*\nabla\mathbf{z}\\&+2i\lambda z_\uparrow^*z_\downarrow\nu\nabla_-\nu+2i\lambda z_\downarrow^*z_\uparrow\nu\nabla_+\nu\\&+2i\lambda\nu^2\mathbf{z}^*\nabla\cdot\sigma\mathbf{z}-\mu\nu^2+\frac{1}{2}g\nu^4\Big].\end{aligned} \tag{4}$$

Here, $\nabla_\pm\equiv\nabla_x\pm i\nabla_y$. It is a nonlinear σ model modified by SOC. The nonlinear effects caused by the particle-particle interactions are, for the most part, contained in the constraint $|\mathbf{z}|=1$.

III. DISPERSION FOR EXCITATIONS OVER PLANE-WAVE CONDENSATES

The mean-field approximation cannot distinguish whether the condensates have PW or SP order when $g_{\uparrow\uparrow}=g_{\downarrow\downarrow}=g_{\uparrow\downarrow}$. However, after taking into account Gaussian fluctuations, the authors of Ref. [26] find that the condensates with PW order have a lower ground-state energy than those with SP order. So they predict that the condensates prefer PW to SP order.

Based on their result, in this paper we study the properties of the excitations over condensates with PW order. We choose a coordinate system such that the momentum of the condensates is located on the k_x axis with coordinates $k_x=-\lambda$ and $k_y=0$. The corresponding wave function of the condensate is $\Psi_0(\mathbf{r},\tau)=\nu_0\frac{1}{\sqrt{2}}e^{-i\lambda x}(1,1)^T$ (see Eq. (3) in Ref. [13]), where $\varphi_\mathbf{k}=\arg(k_x+ik_y)=\pi$ and ν_0 is the square root of the density of the condensates n_0, $\nu_0=\sqrt{n_0}$. Then the boson fields can be written as

$$\Psi(\mathbf{r},\tau)=[\nu_0+\delta\nu(\mathbf{r},\tau)]e^{-i\lambda x+i\theta}\begin{pmatrix}\cos\left(\frac{\pi}{4}+\phi\right)e^{-i\frac{\xi}{2}}\\ \sin\left(\frac{\pi}{4}+\phi\right)e^{i\frac{\xi}{2}}\end{pmatrix}. \tag{5}$$

Here, $\delta\nu(\mathbf{r},\tau)$ are density fluctuations over the condensates and $\theta(\mathbf{r},\tau)$ are the corresponding phase fluctuations, while $\phi(\mathbf{r},\tau)$ are spin fluctuations and $\xi(\mathbf{r},\tau)$ are the corresponding phase fluctuations.

We substitute the above expression into the Lagrangian, Eq. (4), and expand it in orders of fluctuations. In the zeroth order, the Lagrangian for the condensates is $L_0=\beta V(-(\mu+\lambda^2)n_0+\frac{1}{2}gn_0^2)$, which takes a minimum value at $n_0=\frac{\mu+\lambda^2}{g}$. This means that it must satisfy $\mu+\lambda^2>0$ for bosons to condense. Interestingly, for negative μ the bosons do not condense in the case without SOC, but they do with the inclusion of large enough SOC.

The Lagrangian for fluctuations of quadratic order after taking Fourier transformation is $L'=\int d\mathbf{k}d\omega V(-\mathbf{k},-\omega)^TM(\mathbf{k},\omega)V(\mathbf{k},\omega)$, with $V(\mathbf{k},\omega)\equiv(\theta(\mathbf{k},\omega),\xi(\mathbf{k},\omega),\phi(\mathbf{k},\omega),\delta\nu(\mathbf{k},\omega))^T$, and

$$M(\mathbf{k},\omega)=\begin{pmatrix}n_0k^2 & -in_0\lambda k_y & 0 & -\omega\sqrt{n_0}\\ in_0\lambda k_y & n_0\left(\frac{k^2}{4}+\lambda^2\right) & -n_0\left(\frac{\omega}{2}-2i\lambda k_x\right) & 0\\ 0 & n_0\left(\frac{\omega}{2}-2i\lambda k_x\right) & n_0(k^2+4\lambda^2) & 2i\sqrt{n_0}\lambda k_y\\ \omega\sqrt{n_0} & 0 & -2i\sqrt{n_0}\lambda k_y & k^2+m_\nu\end{pmatrix}, \tag{6}$$

where $k=|\mathbf{k}|$. Above, $\delta\nu$ is gapped with mass $m_\nu=2gn_0$. In the mean-field approximation $m_\nu=2(\mu+\lambda^2)$, and it does not depend on g. Matrix elements with ω reflect the Heisenberg uncertainty relation between amplitude and phase fluctuations. The $\delta\nu,\theta$ section is composed of density-wave (DW) excitations. It becomes soft for a momentum around $\mathbf{k}=(0,0)$, The ϕ,ξ section is composed of spin-wave (SW) excitations. It goes soft around $\mathbf{k}=(2\lambda,0)$. Differently from the case without SOC, the two sections $\delta\nu,\theta$ and ϕ,ξ are bound together by SOC.

We must emphasize that the fluctuations are over condensates with momentum $\mathbf{k}=(-\lambda,0)$. This means that the momenta of the above fluctuations have been shifted by $-\lambda$ in the k_x direction. The realistic momenta for the soft modes are around $(\pm\lambda,0)$. In this paper, we use the relative momentum unless otherwise specified.

A. Overall band structure

The dispersion relations of the fluctuations are determined by the poles of the corresponding propagators. These are just the solutions of the equation $\mathrm{Det}M(\mathbf{k},\omega_{\mathbf{k}}) = 0$, where the analytic continuum $i\omega \to \omega_{\mathbf{k}} + i\delta$ has been taken. It has

$$\omega_{\mathbf{k}}^4 + b\omega_{\mathbf{k}}^3 + c\omega_{\mathbf{k}}^2 + d\omega_{\mathbf{k}} + e = 0. \tag{7}$$

Here, for simplicity, the detailed expression for the coefficient is omitted. The four roots of this quartic equation are the same as the solutions of the following two quadratic equations:

$$\omega_{\mathbf{k}}^2 + f_{1+}\omega_{\mathbf{k}} + f_{2+} = 0, \quad \omega_{\mathbf{k}}^2 + f_{1-}\omega_{\mathbf{k}} + f_{2-} = 0, \tag{8}$$

$$\begin{aligned} f_{1\pm} &= \frac{b \pm \sqrt{8y + b^2 - 4c}}{2}, \\ f_{2\pm} &= y \pm \frac{by - d}{\sqrt{8y + b^2 - 4c}}, \end{aligned} \tag{9}$$

where y is any real solution of the cubic equation,

$$8y^3 - 4cy^2 + (2bd - 8e)y + e(4c - b^2) - d^2 = 0. \tag{10}$$

By careful analysis, we find that if $\omega_{\mathbf{k}}$ is a solution of the first Eq. (8), then $-\omega_{-\mathbf{k}}$ is also a solution of the second Eq. (8). This particle-hole symmetry is also shown in Bose-Einstein condensates (BECs) without SOC. For BECs without SOC, both the path integral method and the BdG equation method give hole solutions. It is purely a consequence of mathematical disposal, and the "holes" do not exist physically. So solutions with a negative value are neglected. Here with SOC, for the same reason, we neglect solutions with a negative value. Therefore, we only need to solve the second Eq. (8), for it gives a non-negative $\omega(\mathbf{k})$ in the whole momentum space. This is different from the result using the hydrodynamical method [25] but agrees with the Bogoliubov method [26].

The overall band structure for excitations is shown in Fig. 1, where the momenta are in units of λ. The upper band $\omega_{\mathbf{k}}^+$ is gapped for all momenta. The lower band $\omega_{\mathbf{k}}^-$ is somewhat flat in the k_y direction. Because the coefficient e in Eq. (7) is 0 only at momenta (0,0) and $(2\lambda,0)$, $\omega_{\mathbf{k}}^-$ are gapless only at these two points. This means that the condensation lifts the infinite degeneracy of the ring ground state to twofold. The additional breaking of the O(2) symmetry of the ring-shaped ground state makes the dispersion of the Goldstone modes more singular in the infrared limit compared to that without SOC. This is the reason for the flatness of the band, and the flattest direction is the k_y direction.

Because m_ν does not depend on the value of the particle-particle interaction strength g, we find that the dispersion does not rely on the value of g. However, for large enough g there will be three-body losses and the situation is different [33], and our calculation is limited to weak enough interactions g.

The mapping of the lower band $\omega_{\mathbf{k}}^-$ onto the k_x-$\omega_{\mathbf{k}}^-$ plane shows asymmetry about $k_x = \lambda$, as shown clearly in the lower panel in Fig. 1. We must emphasize that the momenta have been shifted by $-\lambda$ in the k_x direction, so the dispersion at the realistic momentum is asymmetric about $k_x = 0$. This is a reflection of the breaking of the time reversal symmetry due to the fact that the bosons are condensed onto a single point of the ring in momentum space. By contrast, for BECs without SOC, $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}$, and the time reversal symmetry is maintained.

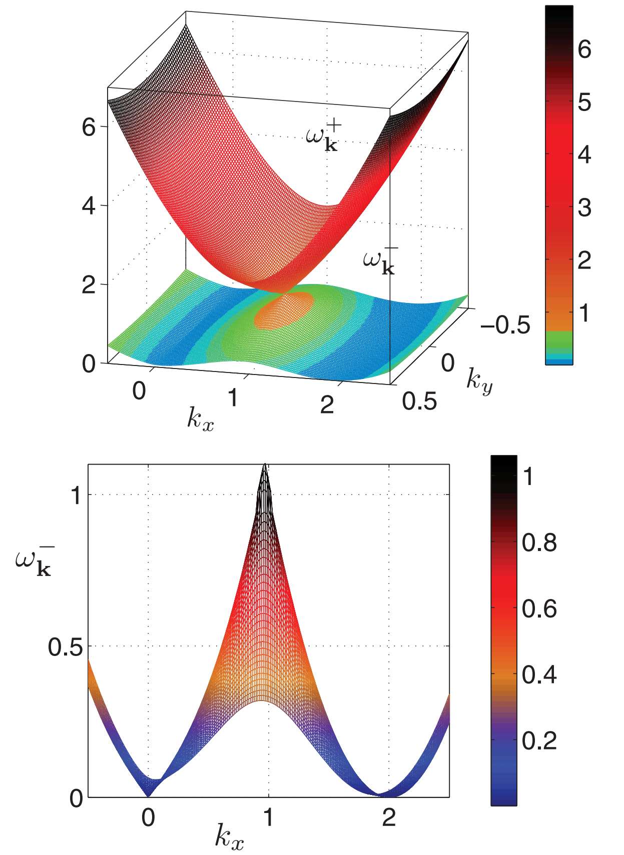

FIG. 1. (Color online) Top: Dispersion of the upper band $\omega_{\mathbf{k}}^+$ and lower band $\omega_{\mathbf{k}}^-$. $\omega_{\mathbf{k}}^+$ is gapped for all momenta. $\omega_{\mathbf{k}}^-$ has some kind of flatness in the k_y direction. There are two soft modes. One is the density wave located around $\mathbf{k} = (0,0)$, and the other is the spin wave located around $\mathbf{k} = (2,0)$. Bottom: Dispersion of $\omega_{\mathbf{k}}^-$ mapped onto the k_x-$\omega_{\mathbf{k}}$ plane. The positions for the soft modes are clearly shown. The momenta have been scaled in units of λ, and $\omega_{\mathbf{k}}^\pm$ in units of $\hbar^2\lambda^2/2m$.

B. Dispersion for soft modes

The excitations become soft around $\mathbf{k} = (0,0)$ and $\mathbf{k} = (2\lambda,0)$, which are DW and SW, respectively. Now we consider the dispersion of these soft modes in the k_x and k_y directions. We choose $k_y = 0$ in the k_x direction and $k_x = 0,2\lambda$ in the k_y direction. The result is shown in Fig. 2.

In the k_x direction with $k_y = 0$, the $\theta,\delta\nu$ and ϕ,ξ sections are decoupled. We only need to solve two quadratic equations to obtain the dispersion. For $\mathbf{k} = (k_x,0)$, the ϕ,ξ section is gapped, while the $\theta,\delta\nu$ section has the dispersion $E_{\mathbf{k}} = |k_x|\sqrt{k_x^2 + m_\nu}$. At small k_x it is linear in k_x. For $\mathbf{k} = (2\lambda + k_x,0)$, the $\theta,\delta\nu$ section is gapped, while the ϕ,ξ section is gapless, with dispersion $E_{\mathbf{k}} \sim k_x^2$. In the k_y direction with $k_x = 2\lambda$ and $k_x = 0$, from numerical calculations we find that the dispersion is quadratic in k_y for both DW and SW excitations. This can also be obtained analytically by calculating the expression of coefficient e in Eq. (7) which equals the multiplication of four roots.

In summary, the dispersion of DWs around $\mathbf{k} = (0,0)$ is linear in k_x at fixed $k_y = 0$, and quadratic in k_y at fixed $k_x = 0$. Note that the dispersion for free bosons, $\varepsilon_0(\mathbf{k}) = (\sqrt{(k_x - \lambda)^2 + k_y^2} - \lambda)^2$, near the condensation point

$\mathbf{k} = (0,0)$ has $\varepsilon_0(k_x,0) \sim k_x^2$ and $\varepsilon_0(0,k_y) \sim \frac{k_y^4}{4\lambda^2}$. We see that the dispersion for excited states over condensates is just approximately the square of the one in the free boson case in these two directions. This is the same as that without SOC. SWs around $\mathbf{k} = (2\lambda,0)$ have an anisotropic quadratic dispersion in the k_x and k_y directions. For a two-component boson system without SOC, the SW section is decoupled from the DW section, which makes the SWs behave like free bosons [32]. However, SOC induces scattering processes with momentum exchange $\mathbf{q} \simeq (\pm 2\lambda,0)$, which couple the DW and SW sections together. As a result, the SW excitations are no longer free bosons. Because of the highly infrared divergence of the SW modes, there is the natural question of the effect of these SW excitations on the stability of the condensates. The result needs detailed calculation of the depletion. We leave this to further work.

IV. STATIC STRUCTURE FACTOR AND FEYNMAN RELATION

A. Static structure factor

The static structure factor $S(\mathbf{k})$ is the Fourier transform of the pair distribution function

$$\frac{N}{V} S(\mathbf{k}) = \int d^d\mathbf{r}\, \langle \delta n(\mathbf{r}) \delta n(\mathbf{0}) \rangle \, e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{11}$$

where V is the volume of the system, N is the total number of bosons, and $\delta n(\mathbf{r})$ is the density fluctuations over the mean density of bosons $\bar{n} = N/V$. It measures the strength of scattering pairs of atoms out of the condensates to form the so-called quantum depletion [34]. There is

$$\frac{N}{V} S(\mathbf{k}) = \langle \delta n(\mathbf{k}) \delta n(-\mathbf{k}) \rangle \simeq 4n_0 \langle \delta \nu(\mathbf{k}) \delta \nu(-\mathbf{k}) \rangle = \frac{2n_0}{\beta} \sum_{n=-\infty}^{\infty} e^{i\omega_n \delta^+} \frac{k^2(k^2+4\lambda^2)^2 - 4\lambda^2 k_y^2 (k^2+4\lambda^2) - k^2(i\omega_n + 4\lambda k_x)^2}{(i\omega_n - \omega_{\mathbf{k}}^+)(i\omega_n + \omega_{-\mathbf{k}}^+)(i\omega_n - \omega_{\mathbf{k}}^-)(i\omega_n + \omega_{-\mathbf{k}}^-)}, \tag{12}$$

where δ^+ is an infinite small positive number. $\omega_n = 2\pi n T$ are the frequencies of bosons at a finite temperature where n is the integer number. $\mathbf{k}$ is the relative momentum against the condensed point $(-\lambda,0)$. We assume weak interactions such that the interaction-induced quantum depletions are small and $n_0 \simeq \frac{N}{V}$.

The expression of $S(\mathbf{k})$ shows that it is symmetric with respect to the k_x and k_y axis. The result with $k_x > 0, k_y > 0$ is shown in Fig. 3. We see that $S(\mathbf{k})$ decreases as the momentum $\mathbf{k}$ becomes small, where the DW becomes soft. And it becomes 0 at the condensed point $\mathbf{k} = \mathbf{0}$. This means that the correlation goes to 0 where the quantum coherence is the largest. $S(\mathbf{k})$ remains finite and featureless as the momentum SW becomes soft.

When $\mathbf{k}$ is small, $S(\mathbf{k})$ is linear in k_x at $k_y = 0$ and quadratic in k_y at $k_x = 0$, as shown in Fig. 3. Like the behavior of the dispersion, in these two directions $S(\mathbf{k})$ at momenta near the condensation point are just the square of the dispersion for free bosons. For a high momentum, the static structure factor goes to unity just as for free bosons.

Now we discuss the change in the static structure factor with a change in the SOC strength. The result is shown in

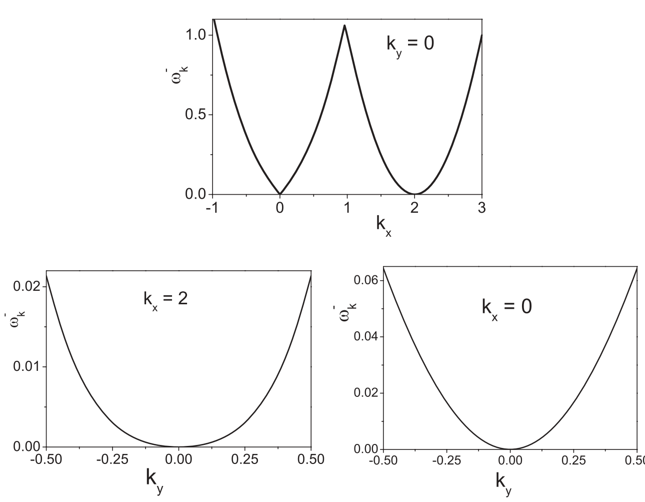

FIG. 2. Top: Dispersion of the lower band $\omega_{\mathbf{k}}^-$ along the k_x direction at $k_y = 0$. Bottom: Dispersion of the lower band $\omega_{\mathbf{k}}^-$ along the k_y direction: (left) $k_x = 2$; (right) $k_x = 0$. We see that around $\mathbf{k} = (2,0)$, the dispersion is quadratic and anisotropic in two directions. And around $\mathbf{k} = (0,0)$, it is linear in the k_x direction and quadratic in the k_y direction. The momenta have been scaled in units of λ, and $\omega_{\mathbf{k}}^-$ in units of $\hbar^2\lambda^2/2m$.

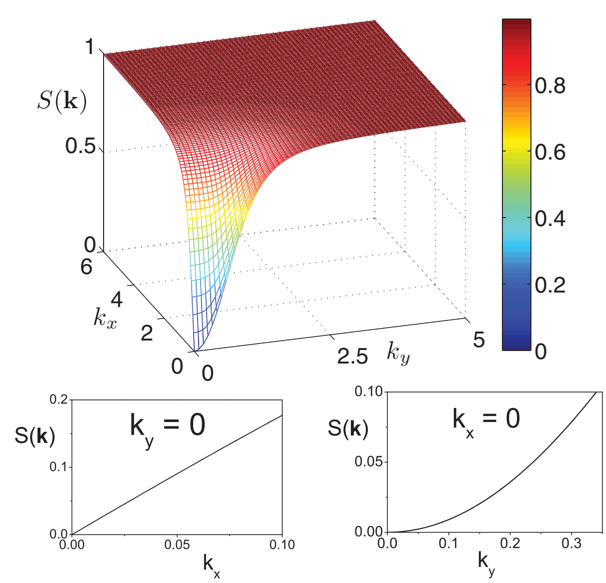

FIG. 3. (Color online) The static structure factor $S(\mathbf{k})$. Momenta have been scaled in units of λ. Top: $S(\mathbf{k})$ goes to 0 at $\mathbf{k} = (0,0)$, where the density-wave modes become soft. It becomes unity at a larger $\mathbf{k}$, which has the same behavior as free particles. Bottom: (Left) $S(\mathbf{k})$ vs k_x at $k_y = 0$; (right) $S(\mathbf{k})/k_y$ vs k_y at $k_x = 0$. We see that $S(\mathbf{k})$ is linear in the k_x direction and quadratic in the k_y direction.

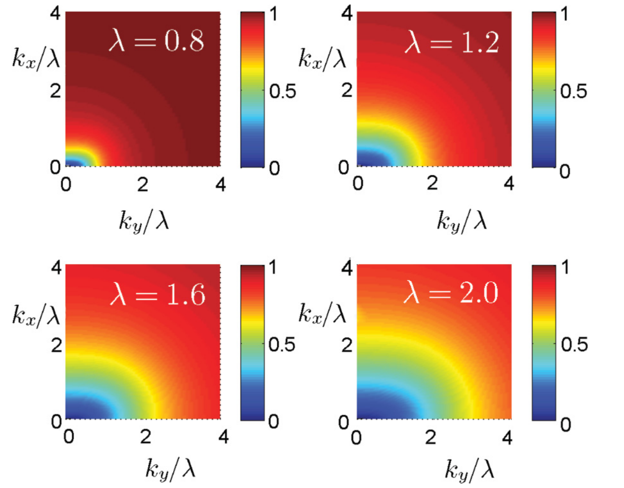

FIG. 4. (Color online) The static structure factor $S(\mathbf{k})$ mapped onto the k_x-k_y plane. The chemical potential $\mu = -0.5$ here. We see that increasing the spin-orbit coupling strength λ reduces the static structure factor and, hence, makes the bosons more coherent.

Fig. 4 for a negative chemical potential and in Fig. 5 for a positive chemical potential. We find that enhancing λ reduces the static structure factor in a larger region of momenta. This is a reflection of the enhancement of the two processes of exciting opposite relative momenta; here the relative momenta are momenta relative to the momentum of condensates [35].

B. Feynman relation

In a one-component boson system, the Feynman relation, Eq. (1), holds true in the whole momentum space [36]. It is a reflection of the f-sum rule [29]. However, a multiple-band structure can make this simple relation exist only at a special momentum where the scattering processes between the lowest band and the condensates dominate [37].

With SOC, the excitations consist of two bands. In addition, the dispersion is highly anisotropic and time reversal symmetry is broken. For weak interactions with $n_0 \simeq \frac{N}{V}$, by putting the

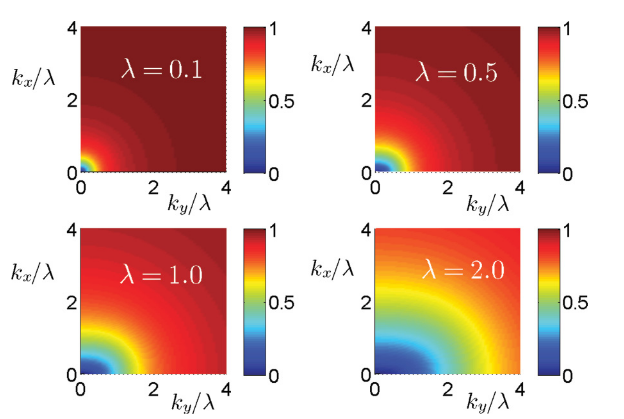

FIG. 5. (Color online) The static structure factor $S(\mathbf{k})$ mapped onto the k_x-k_y plane. The chemical potential $\mu = 0.1$ here. We see that increasing the spin-orbit coupling strength λ reduces the static structure factor and, hence, makes the bosons more coherent.

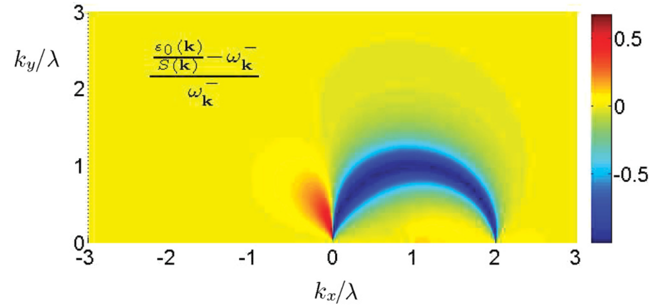

FIG. 6. (Color online) $\frac{\frac{\varepsilon_0(\mathbf{k})}{S(\mathbf{k})} - \omega_{\mathbf{k}}^-}{\omega_{\mathbf{k}}^-}$ is the relative error of the Feynman relation $\omega_{\mathbf{k}}^- = \frac{\varepsilon_0(\mathbf{k})}{S(\mathbf{k})}$ in boson systems with spin-orbit coupling. Here $\varepsilon_0(\mathbf{k}) = (\sqrt{(k_x - \lambda)^2 + k_y^2} - \lambda)^2$ is the dispersion for free bosons. There is symmetry for $k_y \leftrightarrow -k_y$, so only the $k_y > 0$ case is shown in the diagram. We see that the Feynman relation is not satisfied around the "ring" in the momentum space.

dispersion for free bosons $\varepsilon_0(\mathbf{k}) = (\sqrt{(k_x - \lambda)^2 + k_y^2} - \lambda)^2$ in Eq. (1), we find that the Feynman relation does not apply in the whole momentum space (see Fig. 6).

In Fig. 6, the relative error of the Feynman relation, $\frac{\frac{\varepsilon_0(\mathbf{k})}{S(\mathbf{k})} - \omega_{\mathbf{k}}^-}{\omega_{\mathbf{k}}^-}$, is calculated and is mapped onto momentum space. There is symmetry for $k_y \leftrightarrow -k_y$, so only the $k_y > 0$ case is shown. We see that the Feynman relation is not satisfied around the "ring" in the momentum space. The energy of the free bosons is 0 on the ring in the momentum space. However, due to condensation of bosons onto a single point of the ring, the energy of excitations on the ring is increased except at the two points (0,0) and (2λ,0). And the static structure factor is 0 only at the condensed point (0,0). So the naive Feynman relation, Eq. (1), is clearly no longer applicable. In the same reason, we expect that the relation is not satisfied for general particle-particle interactions.

Furthermore, by setting the SOC strength λ to 0 in the Lagrangian, Eq. (4), it can be calculated easily that the static structure factor $S(\mathbf{k})$ and dispersion of excitations $\omega_{\mathbf{k}}$ satisfy the Feynman relation, Eq. (1). So the violation of the Feynman relation found in this work is not due to the fact that we are dealing with a two-component system. It is purely an effect of the existence of SOC.

V. DISCUSSION

We calculate the dispersion relation $\omega_{\mathbf{k}}$ and static structure factor $S(\mathbf{k})$ for Bose-Einstein condensation with SOC for two dimensions at zero temperature. SOC leads to a gapped band $\omega_{\mathbf{k}}^+$ and another band $\omega_{\mathbf{k}}^-$ with two anisotropic gapless modes. One is the DW, with a linear dispersion in the direction of the vector for PWs and a quadratic dispersion perpendicular to this direction. It is more divergent in the infrared limit than in cases without SOC. The other is the SW, with an anisotropic quadratic dispersion in both of these directions. The anisotropy is due to breaking of the O(2) symmetry of the ground state. Time reversal symmetry is also broken for the dispersion because of condensation onto a single momentum with a finite value. The static structure factor $S(\mathbf{k})$ becomes

0 at the momentum where the DW becomes soft, while it remains finite at the momentum of the spin excitation. We also see from the static structure factor that SOC enhances the coherence of excitations out of condensates with opposite relative momenta; here the relative momenta are momenta relative to the momentum of condensates. It is interesting to study the corresponding quantum squeezed states [38,39]. Near the condensation momentum and on the k_x and k_y axes, the dependences of $\omega_{\mathbf{k}}$ and $S(\mathbf{k})$ on the momentum are the squares of the dispersion for free bosons. This relation is just the same as that without SOC. It may be a general characteristic of BECs.

In the presence of SOC, the dispersion of the excitations is quite complex. And the Feynman relation, which relates the energy of excitations, the corresponding static structure factor in the condensed systems, and the dispersion of free bosons, is not satisfied in the whole momentum space. Such a violation should be also expected for general particle-particle interactions, because the condensation induces an increase in the degeneracy of the ground state.

Experimentally, Bragg spectroscopy can be used to measure the static structure factor and also the dispersion of excitations [33,40,41]. So the result in this paper can be checked. The behavior of the static structure factor when the strength of the SOC is changed and, also, the g independence of the dispersion relation with weak enough particle-particle interactions g can be checked as well.

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