Stability of a two-dimensional homogeneous spin-orbit-coupled boson system

Pei-Song He,[1,2] Wen-Long You,[3] and Wu-Ming Liu[2]
[1]*School of Science, Beijing Technology and Business University, Beijing 100048, China*
[2]*Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China*
[3]*School of Physical Science and Technology, Soochow University, Suzhou, Jiangsu 215006, China*
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We investigate the condensation of a two-dimensional homogeneous spin-orbit-coupled boson system at zero temperature. We prove that the condensate is stable although the spin-orbit coupling makes the momentum distribution of depletion more divergent in the infrared limit. The stability of the system depends solely on the infrared behavior of the density waves, while the spin waves play a nonessential role. The condensate fraction is a decreasing function of the ratio of spin-orbit coupling to the square root of the boson density. The peak of the momentum distribution of depletion stretches anisotropically when the strength of spin-orbit coupling increases.

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I. INTRODUCTION

Spin-orbit coupling (SOC), inherited from the interaction between a quantum particle's spin and its motion, has been studied extensively in electronic systems and leads to a rich phenomenology, such as spin Hall effect [1,2], topological insulator [3], and Majorana fermion [4]. Recently, through putting neutral bosonic atoms into a synthetic non-Abelian gauge potential, a system of effective spin-orbit coupled ultracold bosons with equal Rashba [5] and Dresselhaus [6] strength is realized experimentally [7]. Since there is no analog of such a system in solid state materials, it has not been well investigated before. It brought much excitement both theoretically and experimentally, and some interesting phenomena have been theoretically predicted in this many-body system, such as half-Skyrmion and various types of vortex structures [8–17], unconventional phase transitions [18–21], and dynamics [22,23]. Some extensions of the system are also proposed [24–27].

For a two-dimensional homogeneous free bosonic gas with a Rashba SOC, the ground state consists of a circle in the momentum space, which is macroscopically degenerate. When s-wave scattering between particles is considered, the authors in Ref. [28] found at mean-field level that the finite degeneracy is lifted by the interparticle interactions. The ground state is a Bose-Einstein condensate (BEC) with either plane wave (PW) or striped phase (SP), depending on whether interactions between same components of bosons dominate over those between different components. The one with PW order has bosons condensed on one point of the circle in the momentum space, while in the case of SP order bosons are condensed on two points with opposite momenta. When quantum fluctuations are considered, the stability of the condensate in the thermodynamic limit needs to be clarified.

In ultracold atom experiments, physicists prepare BECs by confining bosonic atoms in a trap and observe the condensate only for a short period of time. Theoretically, the Pitaevskii-Stringari theorem, which proves the stability of a condensate in two dimensions at zero temperature, only applies in systems without velocity dependent potential [29,30]. Hence, it is not applicable in boson systems with SOC, which can be considered as a vectorial gauge potential coupled with the movement of bosons. The methods using Gross-Pitaevskii (GP) equation [28] to take into account quantum fluctuations have found stable condensate, but its solution by imaginary time evolution has the shortage that the obtained state may be located at one of the local minima of the free energy. Using Gaussian corrections, the authors in Ref. [31] found finite quantum depletions in two dimensions at zero temperature. Yet, since no explicit expression for the spectra of fluctuations can be obtained, the spectra in the infrared region, which are crucial in proving the finiteness of the depletions, are approximately expanded in momentum.

In this work, based on a nonlinear σ model (NLSM) scheme, which is able to give a definite expression of the dispersion of fluctuations, and allows to investigate the finiteness of the quantum depletions using an inequality. Consequently, we prove rigorously that the infrared contributions to the quantum depletions are finite. We also find that the SOC term augments the phase space of low-energy eigenstates of free boson Hamiltonian, which boosts the effects of interparticle interactions on creating quantum depletions. In other words, the condensate fraction is reduced as the SOC strength becomes larger, and the peak of the momentum distribution of depletion stretches anisotropically when the SOC amplitude increases. Besides, since the strength of SOC is dimensionful, the role played by SOC term is affected with variations of the total boson density of the system, and the condensate fraction decreases when the boson density is reduced.

This paper is organized as follows. Sec. II gives the model and the dispersion relation of excitations. We prove the condensate is stable in Sec. III. The condensate fraction is obtained in Sec. IV. Section V gives the conclusion of this paper.

II. EXTENDED NONLINEAR σ MODEL

The Lagrangian of a two-dimensional homogeneous boson system with a Rashba SOC reads

$$\mathcal{L} = \int_0^\beta d\tau d^2\mathbf{r}\Psi^*(\mathbf{r},\tau)\left(\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 + 2i\lambda\nabla\cdot\sigma - \mu\right) \times \Psi(\mathbf{r},\tau) + \frac{1}{2}\int_0^\beta d\tau d^2\mathbf{r}[g_{\uparrow\uparrow}n_\uparrow(\mathbf{r},\tau)^2 + g_{\downarrow\downarrow}n_\downarrow(\mathbf{r},\tau)^2 + 2g_{\uparrow\downarrow}n_\uparrow(\mathbf{r},\tau)n_\downarrow(\mathbf{r},\tau)], \tag{1}$$

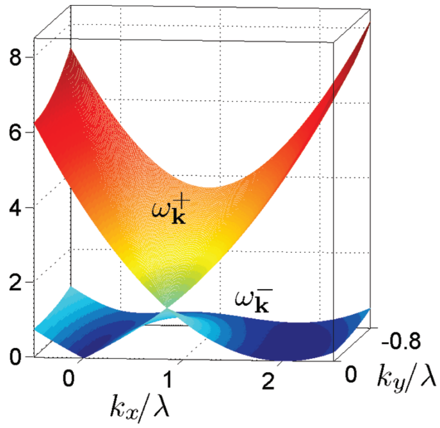

FIG. 1. (Color online) Dispersion relation of the fluctuations over condensate, where the momenta are relative to the condensed point $\mathbf{k} = (-\lambda,0)$. The dispersions are equal with momentum (k_x,k_y) and $(k_x,-k_y)$, so only those with $k_y \leqslant 0$ are shown. $\omega_{\mathbf{k}}^{+}$ composes the upper band and is gapped in the whole momentum space. $\omega_{\mathbf{k}}^{-}$ becomes soft for momentum around $\mathbf{k} = (0,0)$, which is linear in k_x and quadratic in k_y, and around $\mathbf{k} = (2\lambda,0)$, which is quadratic both in k_x and k_y direction.

where $\Psi(\mathbf{r},\tau) = [\psi_\uparrow(\mathbf{r},\tau),\psi_\downarrow(\mathbf{r},\tau)]^T$ is a two-component boson field with pseudospin $\uparrow, \downarrow$. τ is the imaginary time in the path-integral scheme, and $\beta = \frac{\hbar}{k_B T}$ with k_B the Boltzmann constant. $n_\sigma(\mathbf{r},\tau) = \psi_\sigma^*(\mathbf{r},\tau)\psi_\sigma(\mathbf{r},\tau)$ is the density of bosons with spin σ at coordinates $\mathbf{r},\tau$, where $\sigma = \uparrow, \downarrow$. λ is the strength of SOC that couples boson's spin and its momentum. $g_{\sigma\sigma'}$ is strength of s-wave scattering between bosons with spin σ and σ'.

Mean-field analysis finds $g_{\uparrow\uparrow} = g_{\downarrow\downarrow} = g_{\uparrow\downarrow}$ is a transition point between BEC with PW and SP order. By including Gaussian fluctuations, it is found that the condensate prefers PW order [31]. In this work, we study its stability in NLSM scheme, which proves successful for cases with spontaneous breaking of continuous symmetry [32,33]. We set the condensed momentum as $\mathbf{k}_0 \equiv (-\lambda,0)$. The dispersion relations of the excited states $\omega_{\mathbf{k}}$ are solved in our previous work [34], which satisfy a quartic equation

$$\omega_{\mathbf{k}}^4 + b\omega_{\mathbf{k}}^3 + c\omega_{\mathbf{k}}^2 + d\omega_{\mathbf{k}} + e = 0, \tag{2}$$

where the coefficients are $b = 8\lambda k_x, c = -[16\lambda^4 + 8\lambda^2 k^2 - 16\lambda^2 k_x^2 + 8\lambda^2 k_y^2 + 2k^4 + m_\nu k^2]$, $d = -8\lambda k_x[4\lambda^2 k_y^2 + (k^2 + m_\nu)k^2]$, $e = (k^4 - 4\lambda^2 k_x^2)^2 + m_\nu[k_x^2(k^2 - 4\lambda^2)^2 + k^2 k_y^2(k^2 + 4\lambda^2)]$. m_ν, with value $2gn_0$, is the mass of $\delta\nu$ fluctuations. The interparticle interactions enter into the theoretical description through this parameter. The four solutions are labeled as $\omega_{\mathbf{k}}^{+}, -\omega_{-\mathbf{k}}^{+}, \omega_{\mathbf{k}}^{-}, -\omega_{-\mathbf{k}}^{-}$, where $\omega_{\mathbf{k}}^{+}$ and $\omega_{\mathbf{k}}^{-}$ are shown in Fig. 1. The dispersion relations obtained in NLSM scheme have definite expression, it allows using inequality to prove the finiteness of quantum depletions.

III. DEPLETION n_{ex}

Now we discuss whether excitations destroy the condensate at zero temperature. The density of a boson system n is composed of condensate part n_0 and that of excitations n_{ex}. The condition for a stable condensate is $n_0 > 0$, or equally $n_{\mathrm{ex}} < n$. As long as the calculated depletion is not divergent, in principle one can make it smaller than the total density of boson by tuning down the interparticle interactions g, which can be realized by Feshbach resonance in ultracold atom experiments [35]. Therefore, in the following we only need to prove that quantum depletion n_{ex} is not divergent.

The depletion n_{ex} is obtained following the method in Ref. [32]. The boson fields are written as a sum of condensate and fluctuations, that is $\Psi(\mathbf{r},\tau) = \Psi_0(\mathbf{r},\tau) + \delta\Psi(\mathbf{r},\tau)$, and there is

$$\begin{aligned} n_{\mathrm{ex}} &= \frac{1}{V}\sum_{\mathbf{k}} n_{\mathrm{ex}}(\mathbf{k}) = \frac{1}{\beta V}\sum_{\mathbf{k},n} e^{i\omega_n\delta^+} n_{\mathrm{ex}}(\mathbf{k},\omega_n) \\ &= \frac{1}{\beta V}\sum_{\mathbf{k},n} e^{i\omega_n\delta^+}\langle\delta\Psi^*(\mathbf{k},i\omega_n)\delta\Psi(\mathbf{k},i\omega_n)\rangle, \end{aligned} \tag{3}$$

where $\delta\Psi(\mathbf{k},i\omega_n) = \sum_{\mathbf{r}}\int_0^\beta d\tau e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_n\tau}\delta\Psi(\mathbf{r},\tau)$. $\omega_n = 2\pi nT$ is the Matsubara frequency with n an integer ranging from $-\infty$ to ∞. δ^+ is a positive infinitesimal. The fluctuations $\delta\Psi(\mathbf{r},\tau)$ are expanded up to linear order of $\delta\nu(\mathbf{r},\tau),\phi(\mathbf{r},\tau),\theta(\mathbf{r},\tau),\xi(\mathbf{r},\tau)$ as

$$\delta\Psi(\mathbf{r},\tau) \simeq \nu_0\frac{e^{-i\lambda x}}{\sqrt{2}}\begin{pmatrix} -\phi - i\frac{\xi}{2} + i\theta \\ \phi + i\frac{\xi}{2} + i\theta \end{pmatrix} + \delta\nu\frac{e^{-i\lambda x}}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4}$$

This is the *only* approximation taken in this work. It has been proved successful in obtaining depletion in one-component BEC [32]. It is also verified in studying the two-component boson systems without SOC.

Putting the fluctuations in Eq. (4) into Eq. (3), we obtain the depletion as

$$\begin{aligned} n_{\mathrm{ex}} &= \frac{1}{V}\sum_{\mathbf{k}} n_{\mathrm{ex}}(\mathbf{k}) = \frac{1}{V}\sum_{\mathbf{k}} n_{\mathrm{ex}}(\mathbf{k}+\mathbf{k}_0) \\ &= \frac{1}{\beta V}\sum_{\mathbf{k},n} e^{i\omega_n\delta^+}\Big\{ n_0\Big[\frac{1}{4}\langle\xi(-\mathbf{k},-\omega_n)\xi(\mathbf{k},\omega_n)\rangle \\ &\quad + \langle\theta(-\mathbf{k},-\omega_n)\theta(\mathbf{k},\omega_n)\rangle + \langle\phi(-\mathbf{k},-\omega_n)\phi(\mathbf{k},\omega_n)\rangle\Big] \\ &\quad + \langle\delta\nu(-\mathbf{k},-\omega_n)\delta\nu(\mathbf{k},\omega_n)\rangle \\ &\quad + \frac{i}{2}n_0[\langle\phi(-\mathbf{k},-\omega_n)\xi(\mathbf{k},\omega_n)\rangle \\ &\quad - \langle\xi(-\mathbf{k},-\omega_n)\phi(\mathbf{k},\omega_n)\rangle] \\ &\quad + i\sqrt{n_0}[\langle\delta\nu(-\mathbf{k},-\omega_n)\theta(\mathbf{k},\omega_n)\rangle \\ &\quad - \langle\theta(-\mathbf{k},-\omega_n)\delta\nu(\mathbf{k},\omega_n)\rangle]\Big\}. \end{aligned} \tag{5}$$

In the first line of the above equation, we have used the fact that the summation over momentum is invariant by changing variable $\mathbf{k}$ to $\mathbf{k}+\mathbf{k}_0$. $\langle\cdot\rangle$ are Green's functions for the corresponding fields. The approximation we take for the fluctuations, as in Eq. (4), results in bare Green's functions, while the omitted higher-order terms have the effect of dressing them up. For weak interactions g, we expect it is qualitatively correct using bare propagators.

We derive Green's functions in Eq. (5) as in Ref. [32], and obtain $n_{\rm ex}$ as

$$n_{\rm ex} = \frac{1}{\beta V}\sum_{\mathbf{k},n}\frac{e^{i\omega_n\delta^+}[(i\omega_n)^3F_3+(i\omega_n)^2F_2+(i\omega_n)F_1+F_0]}{(i\omega_n-\omega_{\mathbf{k}}^+)(i\omega_n+\omega_{-\mathbf{k}}^+)(i\omega_n-\omega_{\mathbf{k}}^-)(i\omega_n+\omega_{-\mathbf{k}}^-)}, \tag{6}$$

where the coefficients are $F_0 = 16\lambda^4k^2 - 16\lambda^4k_y^2 + 8m_v\lambda^4 + 16\lambda^3k_xk_y^2 + 12\lambda^2k^4 - 16\lambda^2k^2k_x^2 - 8\lambda^2k^2k_y^2 + 8m_v\lambda^2k^2 - 8m_v\lambda^2k_x^2 - 2m_v\lambda^2k_y^2 + 4\lambda k^4k_x + 4m_v\lambda k^2k_x + 2k^6 + 1.5m_vk^4$, $F_1 = 16\lambda^4 + 8\lambda^2k^2 - 16\lambda^2k_x^2 + 8\lambda^2k_y^2 - 4m_v\lambda k_x - 8\lambda k_xk^2 + 2k^4 + m_vk^2, F_2 = 4\lambda^2 + 12\lambda k_x + 2k^2 + 0.5m_v, F_3 = -2$. At zero temperature, the sum over ω_n in Eq. (6) becomes an integral, which can be evaluated as the complex line integral, leading to $n_{\rm ex} = n_{\rm ex}^{(0)} + n_{\rm ex}^{(1)} + n_{\rm ex}^{(2)} + n_{\rm ex}^{(3)}$ with

$$\begin{aligned}
n_{\rm ex}^{(0)} &= \frac{1}{V}\sum_{\mathbf{k}}F_0(\omega_{\mathbf{k}}^+ + \omega_{-\mathbf{k}}^+ + \omega_{\mathbf{k}}^- + \omega_{-\mathbf{k}}^-)/K,\\
n_{\rm ex}^{(1)} &= \frac{1}{V}\sum_{\mathbf{k}}F_1(\omega_{\mathbf{k}}^+\omega_{\mathbf{k}}^- - \omega_{-\mathbf{k}}^+\omega_{-\mathbf{k}}^-)/K,\\
n_{\rm ex}^{(2)} &= \frac{1}{V}\sum_{\mathbf{k}}F_2[\omega_{\mathbf{k}}^+\omega_{\mathbf{k}}^-(\omega_{-\mathbf{k}}^+ + \omega_{-\mathbf{k}}^-)\\
&\quad + \omega_{-\mathbf{k}}^+\omega_{-\mathbf{k}}^-(\omega_{\mathbf{k}}^+ + \omega_{\mathbf{k}}^-)]/K,\\
n_{\rm ex}^{(3)} &= \frac{1}{V}\sum_{\mathbf{k}}F_3\{\omega_{-\mathbf{k}}^+\omega_{-\mathbf{k}}^-[\omega_{\mathbf{k}}^+ + \omega_{\mathbf{k}}^-][\omega_{-\mathbf{k}}^+ + \omega_{-\mathbf{k}}^-]\\
&\quad + \omega_{-\mathbf{k}}^+\omega_{-\mathbf{k}}^-[\omega_{\mathbf{k}}^+\omega_{\mathbf{k}}^- + \omega_{-\mathbf{k}}^+\omega_{-\mathbf{k}}^-]\\
&\quad + \omega_{\mathbf{k}}^+\omega_{\mathbf{k}}^-[(\omega_{-\mathbf{k}}^+)^2 + (\omega_{-\mathbf{k}}^-)^2]\}/K,
\end{aligned} \tag{7}$$

where the denominator $K = (\omega_{\mathbf{k}}^+ + \omega_{-\mathbf{k}}^+)(\omega_{\mathbf{k}}^+ + \omega_{-\mathbf{k}}^-)(\omega_{\mathbf{k}}^- + \omega_{-\mathbf{k}}^+)(\omega_{\mathbf{k}}^- + \omega_{-\mathbf{k}}^-)$.

The first three factors in denominator K are finite for any momentum $\mathbf{k}$, while the fourth one, $\omega_{\mathbf{k}}^- + \omega_{-\mathbf{k}}^-$, approaches 0 for $\mathbf{k} \simeq \mathbf{0}$, where DWs locate. Yet, this term is finite at momentum around $(2\lambda,0)$, where SWs locate, since the excitations of momenta around $(-2\lambda,0)$ are gapped. Therefore, the infrared stability of the condensate is not related to SWs, and it depends solely on DWs. The SWs contribute to depletion in a nonessential way, while in SWs in two-component BEC without SOC give zero contributions to depletion. This difference is an effect of SOC.

In addition, the numerators of $n_{\rm ex}^{(1)}, n_{\rm ex}^{(2)}, n_{\rm ex}^{(3)}$ approach zero for $\mathbf{k} \simeq \mathbf{0}$, while that of $n_{\rm ex}^{(0)}$ is finite. So $n_{\rm ex}^{(0)}$ is the most divergent one in the infrared limit. We consider the low-momentum part of the integral defining $n_{\rm ex}^{(0)}$, that is

$$\begin{aligned}
\tilde{n}_{\rm ex}^{(0)} &= \frac{1}{V}\sum_{|\mathbf{k}|<\delta}\tilde{n}_{\rm ex}^{(0)}(\mathbf{k})\\
&= \frac{1}{V}\sum_{|\mathbf{k}|<\delta}F_0(\omega_{\mathbf{k}}^+ + \omega_{-\mathbf{k}}^+ + \omega_{\mathbf{k}}^- + \omega_{-\mathbf{k}}^-)/K,
\end{aligned} \tag{8}$$

where δ is a small positive number. The ratio of $\tilde{n}_{\rm ex}^{(0)}(\mathbf{k})$ to $1/(\omega_{\mathbf{k}}^- + \omega_{-\mathbf{k}}^-)$, that is $F_0(\omega_{\mathbf{k}}^+ + \omega_{-\mathbf{k}}^+ + \omega_{\mathbf{k}}^- + \omega_{-\mathbf{k}}^-)/(\omega_{\mathbf{k}}^+ + \omega_{-\mathbf{k}}^+)(\omega_{\mathbf{k}}^+ + \omega_{-\mathbf{k}}^-)(\omega_{\mathbf{k}}^- + \omega_{-\mathbf{k}}^+)$, is finite in the limit $|\mathbf{k}| \to 0$. Using a finite number C_1 (approximately $m_v/8$) as the maximum of the ratio in region $|\mathbf{k}| < \delta$, then there is an inequality

$$\tilde{n}_{\rm ex}^{(0)} < C_1\frac{1}{V}\sum_{|\mathbf{k}|<\delta}\frac{1}{\omega_{\mathbf{k}}^- + \omega_{-\mathbf{k}}^-}. \tag{9}$$

Since $\omega_{\mathbf{k}}^- + \omega_{-\mathbf{k}}^- \geqslant 2\sqrt{\omega_{\mathbf{k}}^-\omega_{-\mathbf{k}}^-}$ and the four roots of Eq. (2) satisfy $\omega_{\mathbf{k}}^-(-\omega_{-\mathbf{k}}^-)\,\omega_{\mathbf{k}}^+(-\omega_{-\mathbf{k}}^+) = e$, there is

$$\tilde{n}_{\rm ex}^{(0)} < C_1\frac{1}{V}\sum_{|\mathbf{k}|<\delta}\frac{\sqrt{\omega_{\mathbf{k}}^+\omega_{-\mathbf{k}}^+}}{2\sqrt{e}}. \tag{10}$$

Since $\omega_{\mathbf{k}}^+$ is approximately $4\lambda^2$ at $\mathbf{k} \simeq \mathbf{0}$, similar to Eq. (9), there is $\sqrt{\omega_{\mathbf{k}}^+\omega_{-\mathbf{k}}^+} < 4\lambda^2C_2$ for small momentum, where $4\lambda^2C_2$ is defined as the maximum of $\sqrt{\omega_{\mathbf{k}}^+\omega_{-\mathbf{k}}^+}$ for $\mathbf{k}$ small, and C_2 is a finite constant about one. Further, since $e > m_v[8\lambda^4k_x^2 + 4\lambda^2k_y^4]$, we obtain

$$\tilde{n}_{\rm ex}^{(0)} < \frac{1}{\sqrt{m_v}}C_1C_2\lambda\frac{1}{V}\sum_{|\mathbf{k}|<\delta}\frac{1}{\sqrt{2\lambda^2k_x^2 + k_y^4}} \tag{11}$$

from Eq. (10). In the thermodynamic limit, it can be further changed as

$$\begin{aligned}
\tilde{n}_{\rm ex}^{(0)} &< \frac{1}{\sqrt{m_v}}C_1C_2\lambda\int_{|\mathbf{k}|<\delta}\frac{d^2\mathbf{k}}{4\pi^2}\frac{1}{\sqrt{2\lambda^2k_x^2 + k_y^4}}\\
&= \frac{\sqrt{2}C_1C_2}{8\pi^2\sqrt{m_v}}\int_{x\simeq0,y\simeq0}\frac{dxdy}{\sqrt{x^2+y^4}}\\
&= \frac{\sqrt{2}C_1C_2}{8\pi^2\sqrt{m_v}}\int_{r\simeq0}dr\int_{-\pi}^{\pi}d\theta\frac{1}{\sqrt{\cos^2\theta + r^2\sin^4\theta}},
\end{aligned} \tag{12}$$

where $\frac{1}{V}\sum_{\mathbf{k}} = \int\frac{d^2\mathbf{k}}{(2\pi)^2}$. The poles of the above integration locate at $(\theta,r) \sim (\pm\frac{\pi}{2},0)$. So the most singular part of the integration is

$$\begin{aligned}
I &\equiv \int_{r\sim0}dr\int_{\theta\sim\pm\frac{\pi}{2}}d\theta\frac{1}{\sqrt{\cos^2\theta + r^2\sin^4\theta}}\\
&\overset{\theta=\pm\frac{\pi}{2}+\alpha}{\simeq} 2\int_{r\sim0}dr\int_{\alpha\sim0}d\alpha\frac{1}{\sqrt{\sin^2\alpha + r^2(1-\sin^2\alpha)^2}}\\
&= 2\int_{r\sim0}dr\int_{\beta\sim0}d\beta\frac{1}{\sqrt{(1-\beta^2)[\beta^2 + r^2(1-\beta^2)^2]}}\\
&< 2\int_{r\sim0}dr\int_{\beta\sim0}d\beta\frac{1}{\sqrt{\frac{1}{4}(\beta^2 + \frac{1}{4}r^2)}}\\
&= 4\int_{r\sim0}dr\int_{\beta\sim0}d\beta\frac{1}{\sqrt{\beta^2 + \frac{1}{4}r^2}}.
\end{aligned} \tag{13}$$

Substituted with $\beta = \ell\sin\phi$ and $\frac{1}{2}r = \ell\cos\phi$, the above integration becomes

$$I < 8\int_{\ell\sim0}\ell d\ell\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\phi\frac{1}{\ell} = 8\pi\int_{\ell\sim0}d\ell. \tag{14}$$

It is finite in the infrared limit. It means the depletion in Eq. (6) is bounded from above.

Now we consider the behavior of depletion in the ultraviolet (UV) limit. For brevity, we write the depletion as $n_{\rm ex} = \frac{1}{V}\sum_{\mathbf{k}} n_{\rm ex}(\mathbf{k}) = \frac{1}{V}\sum_{\mathbf{k}} Q/K$. The dispersions of excitations have the form $\omega_{\mathbf{k}}^{\pm} = [\sqrt{(k_x-\lambda)^2+k_y^2} \pm \lambda]^2 + C^{\pm}(\mathbf{k})m_\nu + O(k^{-1})$ in the UV limit, where $C^{\pm}(\mathbf{k})$ are anisotropic in momentum space satisfying $0 \leqslant C^{\pm}(\mathbf{k}) \leqslant \frac{1}{2}$ and $C^{+}(\mathbf{k}) + C^{-}(\mathbf{k}) = \frac{1}{2}$. By substituting this into Eq. (7), it is easy to find that, in the expression of the numerator Q, the coefficients of terms of order k^8 and k^6 are exactly zero. Besides, since the terms of order k^7 and k^5 in Q have odd symmetry in momentum space, they give zero contributions to the depletion after integration over momenta. So in the UV limit, the depletion $n_{\rm ex}$ is an integral over a function at most of order k^{-4}. Further numerical verifications find $n_{\rm ex}(\mathbf{k})$ is approximately proportional to $|\mathbf{k}|^{-4}$ in the UV limit, and it gives negligible contributions to the quantum depletion.

Therefore, for sufficiently weak interparticle interactions, the condensate is stable in two dimensions at zero temperature. This is the main result of this section.

IV. CONDENSATE FRACTION n_0/n

In Sec. III, the quantum depletion is found to be finite. In this section, we will calculate its value explicit, and study its behavior with variations of the parameters of the system. First, we notice that the SOC strength is dimensionful, with canonical dimension $[\lambda] = [\mathbf{k}] = 1$, so the role played by SOC are related to the value of boson density. Secondly, the introducing of SOC term augments the phase space of low-energy excited states of free boson Hamiltonian from a cake around a point (with dispersion $\epsilon_{\mathbf{k}}^0 = k^2$) in momentum space to a ribbon around a circle with radius equals SOC strength [with dispersion $\epsilon_{\mathbf{k}}^0 = (|\mathbf{k}| - \lambda)^2$], see Fig. 2. The increasing of low-energy phase space effectively boosts the role of interparticle interactions, which create quantum depletions in BEC. For BEC with PW order, bosons condense at point O, see Fig. 2, the many-body excited state with momentum $\mathbf{k}$ is created by scattering between the condensate and the excited states of free boson system with momentum $\mathbf{k}$ and $-\mathbf{k}$ [see Eq. (5)]. So the region $ABA'B'$ will have strong quantum fluctuations. Keeping the energy on point A and A' fixed while varying the SOC strength λ, the lengths of OA and OA' change in proportion to $\sqrt{\lambda}$. At meantime, the lengths of OB and OB' keep constant with variation of SOC strength. According to the above analysis, the momentum dependence of quantum depletions should has a peak with the shape like that of the region $ABA'B'$, and the quantum depletions should be enhanced by increasing the SOC strength.

The condensate density is determined self-consistently by the equation [32]

$$N = Vn_0 + \sum_{\mathbf{k}} n_{\rm ex}(\mathbf{k};\lambda,m_\nu), \tag{15}$$

where N is the total number of bosons, and V is the volume of the system. $n = \frac{N}{V}$ is the density of bosons. $n_{\rm ex}(\mathbf{k};\lambda,m_\nu)$ is the momentum dependence of quantum depletions. Using $\eta_0 \equiv n_0/n$ for condensate fraction, the dimensionless form of Eq. (15) becomes

$$\eta_0 = \frac{1}{1 + 2\tilde{g}\int \frac{d^2\tilde{\mathbf{k}}}{4\pi^2} n_{\rm ex}(\tilde{\mathbf{k}};\tilde{\lambda},1)}, \tag{16}$$

where $\tilde{g},\tilde{\mathbf{k}},\tilde{\lambda}$ are dimensionless quantities with $\tilde{g} = 2mg/\hbar^2, \tilde{\mathbf{k}} = \mathbf{k}/\sqrt{2\tilde{g}n\eta_0}$, and $\tilde{\lambda} = 2m\lambda/(\hbar^2\sqrt{2\tilde{g}n\eta_0})$. We see the SOC strength is scaled by square root of boson density, as dimension analysis indicates. The condensate fraction, obtained from Eq. (16), is shown in Fig. 3 as a function of $\lambda/\sqrt{n}$ for various values of interparticle interactions g. We find that when $\lambda/\sqrt{n}$ increases, the condensate fraction decreases, and the reduced value is approximately a square root function of $\lambda/\sqrt{n}$ in a broad region of $\lambda/\sqrt{n}$. This behavior can be quantitatively explained since the majority of the depletions are contributed by low-energy ones, as shown in Fig. 4, and the low-energy peak of depletions stretches in k_y direction in ratio of about $\sqrt{\lambda/\sqrt{n}}$ and keeps unaffected in the k_x direction, as shown in Fig. 5. The behavior of the depletion

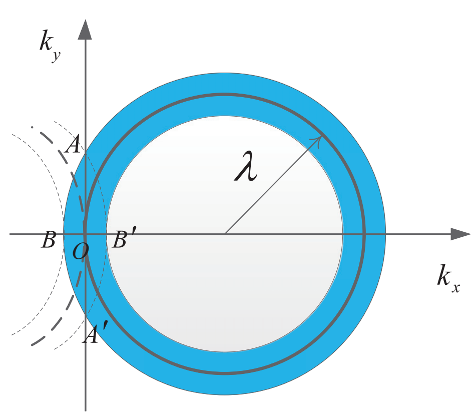

FIG. 2. (Color online) The blue ribbon is the phase space of low-energy excited states of free boson Hamiltonian with SOC. Quantum depletions with momentum within the region $ABA'B'$ have a large population. When the strength of SOC is changed, the lengths of OA and OA' are proportional to $\sqrt{\lambda}$ when the energy of points A and A' is fixed, while those of OB and OB' are unaffected.

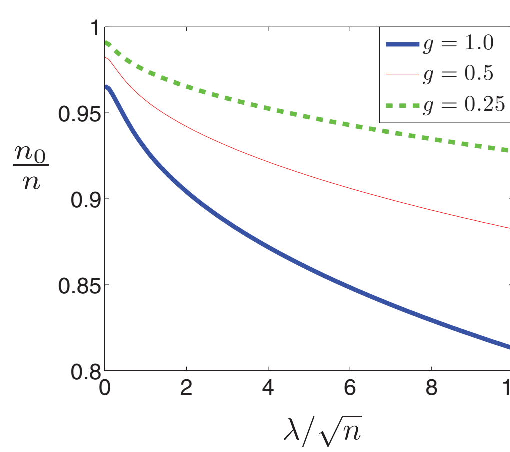

FIG. 3. (Color online) Condensate fraction n_0/n as a function of $\lambda/\sqrt{n}$ with $g = 0.25$ (green dashed), $g = 0.5$ (red thin solid), and $g = 1.0$ (blue thick solid), where the dimensionless parameters g and $\lambda/\sqrt{n}$ are in unit of $\hbar^2/2m$. n_0/n is approximately a square root function of $\lambda/\sqrt{n}$ in a broad region of $\lambda/\sqrt{n}$, except for $\lambda/\sqrt{n}$ near zero.

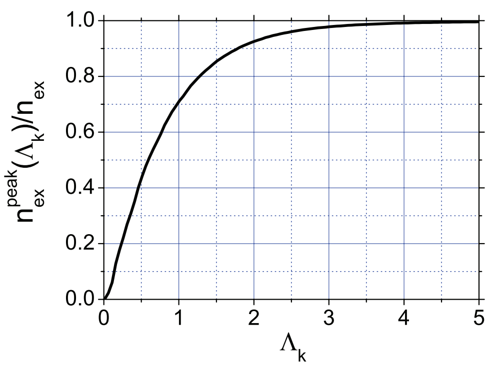

FIG. 4. (Color online) The ratio of the peak part to the total depletion for boson system with $\lambda/\sqrt{n} = 4$, where $\lambda/\sqrt{n}$ and Λ_k are in unit of $\hbar^2/2m$. $n_{\text{ex}}^{\text{peak}}(\Lambda_k)$ is the depletion contributed by excitations with $|\mathbf{k}| \leqslant \Lambda_k$, and n_{ex} is the total depletion of the system.

peak with change of SOC strength obtained is the same to the analysis given above in Fig. 2. Further, the contours of the peaks are also similar to that of the region $ABA'B'$ in Fig. 2. Therefore, we can conclude that the effect of SOC term on quantum depletions is that it enlarges the phase space of low-energy excited states of free boson Hamiltonian, and effectively enhance the role of interparticle interactions on creating quantum depletions.

The effect of boson density on condensate fraction in three-dimensional counterpart is of some difference, since the canonical dimension of interparticle interaction g is not the same as the two-dimensional case. In a three-dimensional boson system with a Rashba SOC in plane, η_0 satisfies

$$1 = \eta_0 + (2\tilde{g}n^{\frac{1}{3}}\eta_0)^{\frac{3}{2}} \int \frac{d^3\tilde{\mathbf{k}}}{(2\pi)^3} n_{\text{ex}}^{(3d)}(\tilde{\mathbf{k}};\tilde{\lambda},1). \tag{17}$$

In Eq. (17), n divides the SOC strength in the form of $\lambda/\sqrt{n}$, and also multiplies the interparticle interaction strength with $gn^{\frac{1}{3}}$. Changing n has two reverse effects on the value of η_0, which is different from the two-dimensional situation. It is interesting to investigate which of the two effects dominates for various values of g,λ, and n.

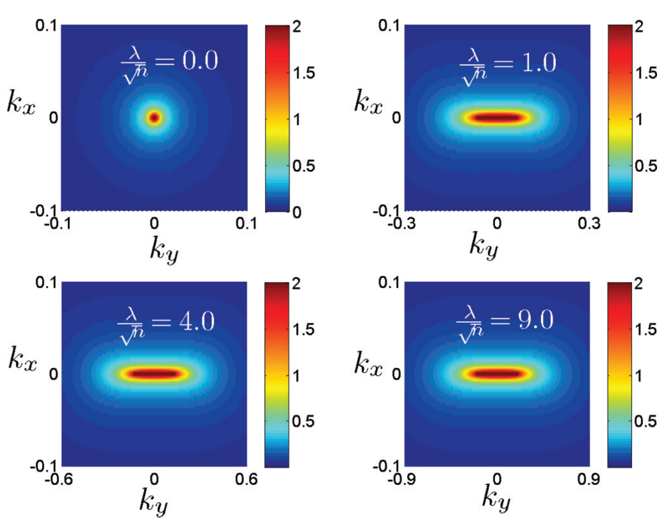

FIG. 5. (Color online) The momentum dependence of depletion $n_{\text{ex}}(\mathbf{k})$ with $\lambda/\sqrt{n} = 0.0,1.0,4.0,9.0$, where k_x,k_y and $\lambda/\sqrt{n}$ are in unit of $\hbar^2/2m$. Around the divergent peak with $\mathbf{k} = \mathbf{0}$, the points with $n_{\text{ex}}(\mathbf{k}) > 2$ are substituted with a value of 2 for clarity. The contours of the three peaks with nonzero $\lambda/\sqrt{n}$ are almost the same by scaling the k_y coordinates with $\sqrt{\lambda/\sqrt{n}}$.

V. DISCUSSION AND CONCLUSION

Based on a NLSM scheme, we prove analytically that a BEC with a Rashba SOC in two dimensions at zero temperature is stable against quantum fluctuations for weak interparticle interactions. We show the infrared stability of the condensate depends solely on DWs, and the SWs give nonessential contributions to the depletions. Furthermore, we find that the SOC term increases the phase space of low-energy eigenstates of the free boson systems, and thus effectively boosts the role of the interparticle interactions on creating quantum fluctuations. In such a case, the depletions increase when the SOC strength is enhanced. Since the strength of SOC is dimensionful, the role played by SOC term is affected with variations of the total boson density of the system. The condensate fraction decreases when the boson density is reduced.

In most experiments of BEC without SOC and not on an optical lattice, the ground-state depletion is of order of 1%, in contrast to that of 90% in liquid helium [36]. Attempts to increase the depletion fraction are limited by the three-body losses, which become more prominent when increasing the boson density or the interparticle interactions strength [37,38]. Our results indicate that a large portion of quantum depletions can be potentially achieved in spin-orbit coupled boson system, by turning up SOC strength and lowering down the boson density, and the three-body losses are kept at low level by holding the interparticle interactions weak. It is also an interesting question whether a boson liquid in free space can be achieved in such a system.

In ultracold atom experiments, the Rashba SOC can be engineered by coupling ^{87}Ru atom's hyperfine states with pairs of lasers [7,22,39,40]. The condensed momentum of the bosons, which represents the order of the condensate, can be measured from the drifting behavior of the atoms in the time-of-flight (TOF) imaging [7]. From the density distributions of atoms with different spin species [7,41], which can be distinguished in TOF imaging by applying Stern-Gerlach magnetic field, the condensate density as well as the momentum dependence of depletions can also be obtained. Our results can then be checked. We hope this work can motivate further work into understanding the behavior of quantum fluctuations in BEC with SOC.

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