Magnetic soliton and soliton collisions of spinor Bose-Einstein condensates in an optical lattice

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We study the magnetic soliton dynamics of spinor Bose-Einstein condensates in an optical lattice which results in an effective Hamiltonian of anisotropic pseudospin chain. A modified Landau-Lifshitz equation is derived and exact magnetic soliton solutions are obtained analytically. Our results show that the time oscillation of the soliton size can be controlled in practical experiment by adjusting of the light-induced dipole-dipole interaction. Moreover, the elastic collision of two solitons is investigated.

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I. INTRODUCTION

Recently, spinor Bose-Einstein condensates (BEC's) trapped in optical potentials have received much attention in both experimental [1–3] and theoretical studies [4]. Spinor BEC's have internal degrees of freedom due to the hyperfine spin of the atoms which liberate a rich variety of phenomena such as spin domains [5] and textures [6]. When the potential valley is so deep that the individual sites are mutually independent, spinor BEC's at each lattice site behave like spin magnets and can interact with each other through both the light-induced and static, magnetic dipole-dipole interactions. These site-to-site dipolar interactions can cause the ferromagnetic phase transition [7,8] leading to a "macroscopic" magnetization of the condensate array and spin-wave-like excitation [7,9] analogous to the spin wave in a ferromagnetic spin chain. For the real spin chain, the site-to-site interaction is caused mainly by the exchange interaction, while the dipole-dipole interaction is negligibly small. For the spinor BEC's in the optical lattice, the exchange interaction is absent. The individual spin magnets are coupled by the magnetic and the light-induced dipole-dipole interactions [9] which are no longer negligible due to the large number of atoms, N, at each lattice site, typically of the order of 1000 or more. Therefore, the spinor BEC's in an optical lattice offer a totally new environment to study spin dynamics in periodic structures. The magnetic soliton excited by the interaction between the spin waves [9] is an important and interesting phenomenon in spinor BEC's. In this paper, we demonstrate that the magnetic soliton and elastic soliton collision are admitted for spinor BEC's in a one-dimensional optical lattice and are controllable by adjusting of the light-induced and the magnetic dipole-dipole interactions.

The Heisenberg model of spin-spin interactions is considered as the starting point for understanding many complex magnetic structures in solids. In particular, it explains the existence of ferromagnetism and antiferromagnetism at temperatures below the Curie temperature. The magnetic soliton [10], which describes localized magnetization, is an important excitation in the Heisenberg spin chain [11–14]. The Haldane gap [15] of antiferromagnets has been reported in integer Heisenberg spin chain. By means of the neutron inelastic scattering [16] and electron spin resonance [17], the magnetic soliton has already been probed experimentally in quasi-one-dimensional magnetic systems. Solitons can travel over long distances with neither attenuation nor change of shape, since the dispersion is compensated by nonlinear effects. The study of solitons has been conducted in as diverse fields as particle physics, molecular biology, geology, oceanography, astrophysics, and nonlinear optics. Perhaps the most prominent application of solitons is in high-rate telecommunications with optical fibers. However, the generation of controllable solitons is an extremely difficult task due to the complexity of the conventional magnetic materials. The spinor BEC's seems an ideal system to serve as a new test ground for studying the nonlinear excitations of spin waves both theoretically and experimentally.

The outline of this paper is organized as follows: In Sec. II the Landau-Lifshitz equation of spinor BEC's in an optical lattice is derived in detail. Next, we obtain the one-soliton solution of spinor BEC's in an optical lattice. The result shows that the time oscillation of the amplitude and the size of soliton can be controlled by adjusting of the light-induced dipole-dipole interaction. We also present that the magnetization varies with time periodically. In Sec. VI, the general two-soliton solution for spinor BEC's in an optical lattice is investigated. Analysis reveals that elastic soliton collision occurs and there is a phase exchange during collision. Finally, our concluding remarks are given in Sec. V.

II. LANDAU-LIFSHITZ EQUATION OF SPINOR BEC's IN AN OPTICAL LATTICE

The dynamics of spinor BEC's trapped in an optical lattice is primarily governed by three types of two-body interactions: spin-dependent collisions characterized by the s-wave scattering length, magnetic dipole-dipole interaction (of the order of Bohr magneton μ_B), and light-induced dipole-dipole interaction adjusted by the laser frequency in experiment. Our starting point is the Hamiltonian describing an $F=1$ spinor condensate at zero temperature trapped in an optical lattice, which is subject to magnetic and light-induced dipole-dipole interactions and is coupled to an external magnetic field via the magnetic dipole Hamiltonian H_B [4–7],

$$H=\sum_{\alpha}\int d\mathbf{r}\hat{\psi}_{\alpha}^{\dagger}(\mathbf{r})\left[-\frac{\hbar^2\nabla^2}{2m}+U_L(\mathbf{r})\right]\hat{\psi}_{\alpha}(\mathbf{r}) + \sum_{\alpha,\beta,\upsilon,\tau}\int d\mathbf{r}d\mathbf{r}'\,\hat{\psi}_{\alpha}^{\dagger}(\mathbf{r})\hat{\psi}_{\beta}^{\dagger}(\mathbf{r}')[U_{\alpha\upsilon\beta\tau}^{coll}(\mathbf{r},\mathbf{r}') + U_{\alpha\upsilon\beta\tau}^{d-d}(\mathbf{r},\mathbf{r}')]\hat{\psi}_{\tau}(\mathbf{r}')\hat{\psi}_{\upsilon}(\mathbf{r})+H_B, \quad (1)$$

where $\hat{\psi}_{\alpha}(r)$ is the field annihilation operator for an atom in the hyperfine state $|f=1,m_f=\alpha\rangle$, $U_L(\mathbf{r})$ is the lattice potential, and the indices $\alpha,\beta,\upsilon,\tau$ which run through the values $-1,0,1$ denote the Zeeman sublevels of the ground state. The parameter $U_{\alpha\upsilon\beta\tau}^{coll}(\mathbf{r},\mathbf{r}')$ describes the two-body ground-state collisions, and $U_{\alpha\upsilon\beta\tau}^{d\text{-}d}(\mathbf{r},\mathbf{r}')$ includes the magnetic dipole-dipole interaction and the light-induced dipole-dipole interaction.

When the optical lattice potential is deep enough there is no spatial overlap between the condensates at different lattice sites. We can then expand the atomic field operator as $\hat{\psi}(\mathbf{r}) = \sum_n\sum_{\alpha=0,\pm1}\hat{a}_{\alpha}(n)\phi_n(\mathbf{r})$, where n labels the lattice sites, $\phi_n(\mathbf{r})$ is the condensate wave function for the nth microtrap, and the operators $\hat{a}_{\alpha}(n)$ satisfy the bosonic commutation relations $[\hat{a}_{\alpha}(n),\hat{a}_{\beta}^{\dagger}(l)]=\delta_{\alpha\beta}\delta_{nl}$. It is assumed that all Zeeman components share the same spatial wave function. If the condensates at each lattice site contain the same number of atoms N, the ground-state wave functions for different sites have the same form $\phi_n(\mathbf{r})=\phi_n(\mathbf{r}-\mathbf{r}_n)$.

In this paper we consider a one-dimensional optical lattice along the z direction, which we also choose as the quantization axis. In the absence of spatial overlap between individual condensates and neglecting unimportant constants, we can construct the effective spin Hamiltonian [7,9] as

$$H=\sum_n\left[\lambda_a'\hat{\mathbf{S}}_n^2-\sum_{l\neq n}\lambda_{nl}\mathbf{S}_n\cdot\mathbf{S}_l+2\sum_{l\neq n}\lambda_{nl}^{ld}\hat{S}_n^z\hat{S}_l^z-\gamma\hat{S}_n\cdot\mathbf{B}\right], \quad (2)$$

where $\lambda_{nl}=2\lambda_{nl}^{ld}+\lambda_{nl}^{md}$, λ_{nl}^{md} and λ_{nl}^{ld} represent the magnetic and light-induced dipole-dipole interactions, respectively. The direction of the magnetic field $\mathbf{B}$ is along the one-dimensional optical lattice, and $\gamma=g_F\mu_B$ is the gyromagnetic ratio. The spin operators are defined as $\mathbf{S}_n=\hat{a}_{\alpha}^{\dagger}(n)\mathbf{F}_{\alpha\upsilon}\hat{a}_{\upsilon}(n)$, where $\mathbf{F}$ is the vector operator for the hyperfine spin of an atom, with components represented by 3×3 matrices in the $|f=1,m_f=\alpha\rangle$ subspace. The first term in Eq. (2) results from the spin-dependent interatomic collisions at a given site, with $\lambda_a'=(1/2)\lambda_a\int d^3r|\phi_n(\mathbf{r})|^4$, where λ_a characterizes the spin-dependent s-wave collisions. The second and third terms describe the site-to-site spin coupling induced by the static magnetic field dipolar interaction and the light-induced dipole-dipole interaction. For $\lambda_{nl}\neq0$, the transfer of transverse excitation from site to site is allowed, resulting in a distortion of the ground-state spin structure. This distortion can propagate and hence generate spin waves along the atomic spin chain. For an optical lattice created by blue-detuned laser beams, the atoms are trapped in the dark-field nodes of the lattice and the light-induced dipole-dipole interaction is very small [8]. However, this small light-induced dipole-dipole interaction induces the amplitude and size of the soliton varying with time periodically as we will show in the following section.

From Hamiltonian (2) we can derive the Heisenberg equation of motion at kth site for the spin excitations. When the optical lattice is infinitely long and the spin excitations are in the long-wavelength limit—i.e., the continuum limit $S_k\to S(z,t)$—we obtain the Landau-Lifshitz equation of a spinor BEC in an optical lattice as follows:

$$\frac{\partial S^x}{\partial t}=\frac{2\lambda}{\hbar}\left[a^2\left(S^y\frac{\partial^2}{\partial z^2}S^z-S^z\frac{\partial^2}{\partial z^2}S^y\right)-4\frac{\lambda^{ld}}{\lambda}S^yS^z\right]+\frac{\gamma BS^y}{\hbar},$$

$$\frac{\partial S^y}{\partial t}=\frac{2\lambda}{\hbar}\left[a^2\left(S^z\frac{\partial^2}{\partial z^2}S^x-S^x\frac{\partial^2}{\partial z^2}S^z\right)+4\frac{\lambda^{ld}}{\lambda}S^zS^x\right]-\frac{\gamma BS^x}{\hbar},$$

$$\frac{\partial S^z}{\partial t}=\frac{2\lambda}{\hbar}\left[a^2\left(S^x\frac{\partial^2}{\partial z^2}S^y-S^y\frac{\partial^2}{\partial z^2}S^x\right)\right], \quad (3)$$

where we assume that all nearest-neighbor interactions are the same—namely, $\lambda_{\langle nl\rangle}=\lambda$, which is a good approximation in the one-dimensional optical lattice for the large lattice constant [19]. In a rotating frame around the z axis with angular frequency $\gamma B/\hbar$ the spin vector S is related to the original one by the transformation

$$S^x=S^{x'}\cos\left(\frac{\gamma B}{\hbar}t\right)+S^{y'}\sin\left(\frac{\gamma B}{\hbar}t\right),$$

$$S^y=S^{y'}\cos\left(\frac{\gamma B}{\hbar}t\right)-S^{x'}\sin\left(\frac{\gamma B}{\hbar}t\right). \quad (4)$$

Thus the Landau-Lifshitz equation (3) in the rotating coordinate system can be written as

$$\frac{\partial}{\partial t}S^x=S^y\frac{\partial^2}{\partial z^2}S^z-S^z\frac{\partial^2}{\partial z^2}S^y-16\rho^2S^yS^z,$$

$$\frac{\partial}{\partial t}S^y=S^z\frac{\partial^2}{\partial z^2}S^x-S^x\frac{\partial^2}{\partial z^2}S^z+16\rho^2S^zS^x,$$

$$\frac{\partial}{\partial t}S^z=S^x\frac{\partial^2}{\partial z^2}S^y-S^y\frac{\partial^2}{\partial z^2}S^x, \quad (5)$$

where $\rho^2=\lambda^{ld}/(4\lambda)$, and the prime is omitted for the sake of pithiness. The dimensionless time t and coordinate z in Eq. (5) are scaled in unit $2\lambda/\hbar$ and a, respectively, where a denotes the lattice constant. Also, the terms including the external magnetic field in Eq. (3) have been eliminated with the help of the transformation.

III. ONE-SOLITON SOLUTION OF SPINOR BEC's IN AN OPTICAL LATTICE

Equation (5) has a form of the Landau-Lifshitz (LL) type which is similar to the LL equation for a spin chain with an easy plane anisotropy [18]. By introducing a particular parameter Huang *et al.* [14] showed that the Jost solutions can

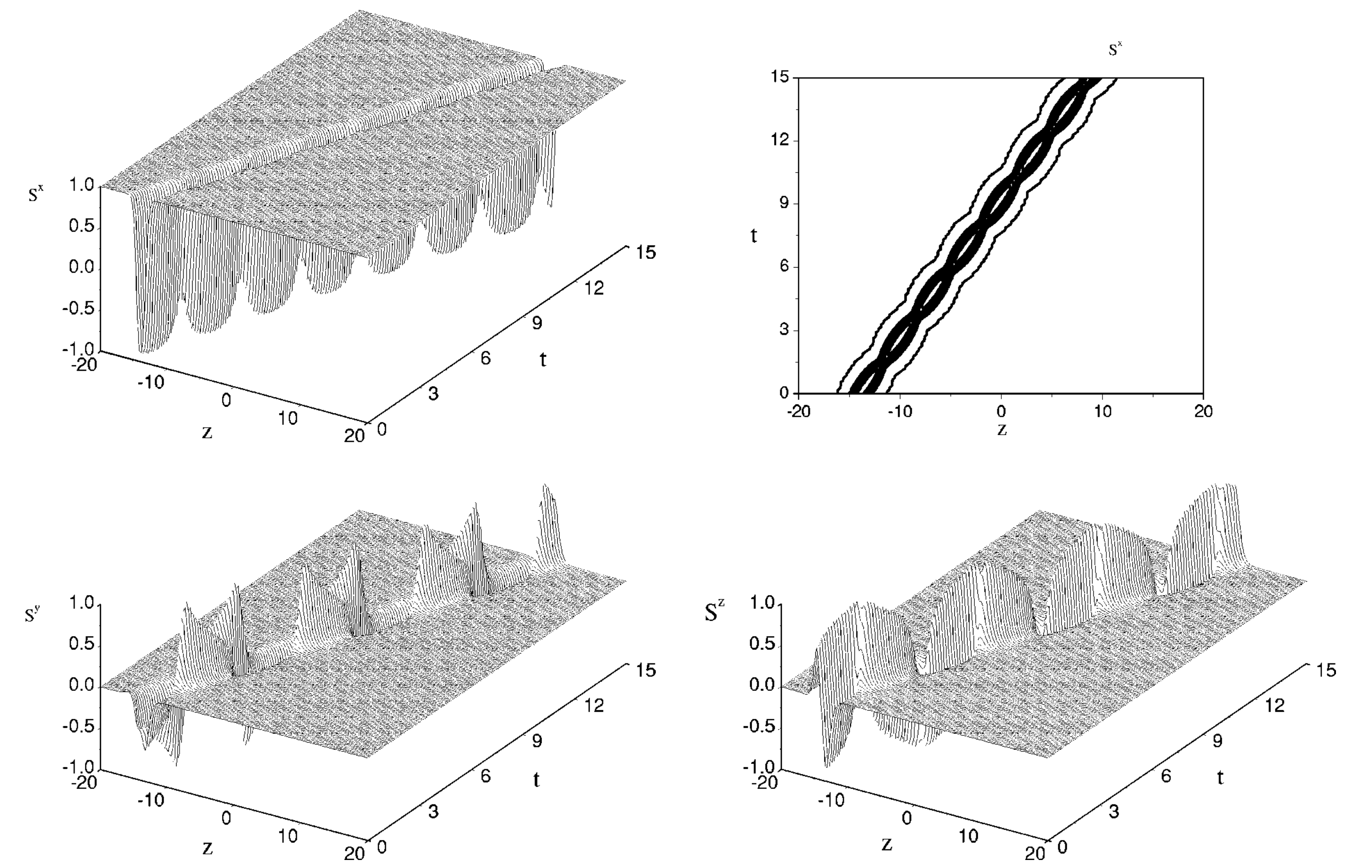

FIG. 1. The amplitude and size of solitons in Eq. (8) vary periodically with time, where $\mu_1=0.45$, $\nu_1=0.7$, $z_1=-14$, $\phi_1=0.5$, and $\rho^2=0.375$. The unit for time t is $2\lambda/\hbar$ and a for space z.

be generated and the Lax equations are satisfied and, moreover, constructed Darboux transformation matrices. An explicit expression of the one-soliton solution in terms of elementary functions of z and t was reported. Here using the similar method in Refs. [13,14] we obtain both the one- and two-soliton solutions (for detail see the Appendix) denoted by $\mathbf{S}(n)$ with $n=1,2$ of Eq. (5) in the following form:

$$S_n^x = 1 - \frac{1}{\Lambda_n}(\chi_{2,n} + 2\chi_{3,n}\sin^2\Phi_n),$$

$$S_n^y = \frac{-1}{\Lambda_n}(\chi_{1,n}\eta_n\cosh\Theta_n\sin\Phi_n + \chi_{2,n}\sinh\Theta_n\cos\Phi_n),$$

$$S_n^z = \frac{1}{\Lambda_n}(\chi_{1,n}\cosh\Theta_n\cos\Phi_n + \chi_{2,n}\eta_n\sinh\Theta_n\sin\Phi_n), \tag{6}$$

where the parameters in the solution are defined by

$$\Lambda_n = \cosh^2\Theta_n + \chi_{3,n}\sin^2\Phi_n,$$

$$\Theta_n = 2\kappa_{4,n}(z - V_n t - z_n),$$

$$\Phi_n = 2\kappa_{3,n}z - \Omega_n t + \phi_n,$$

$$V_n = 2\left(\kappa_{1,n} + \frac{\kappa_{3,n}}{\kappa_{4,n}}\kappa_{2,n}\right),$$

$$\Omega_n = 4(\kappa_{1,n}\kappa_{3,n} - \kappa_{2,n}\kappa_{4,n}), \tag{7}$$

with $\kappa_{1,n}=\mu_n(1+\rho^2/|\zeta_n|^2)$, $\kappa_{2,n}=\nu_n(1-\rho^2/|\zeta_n|^2)$, $\kappa_{3,n}=\mu_n(1-\rho^2/|\zeta_n|^2)$, $\kappa_{4,n}=\nu_n(1+\rho^2/|\zeta_n|^2)$, $\eta_n=(|\zeta_n|^2+\rho^2)/(|\zeta_n|^2-\rho^2)$, $\chi_{1,n}=(2\mu_n\nu_n)/|\zeta_n|^2$, $\chi_{2,n}=(2\nu_n^2)/|\zeta_n|^2$, and $\chi_{3,n}=(4\rho^2\nu_n^2)/(|\zeta_n|^2-\rho^2)^2$. The one-soliton solution—namely, $\mathbf{S}(1)$—is simply that

$$S^x(1) = S_1^x, \quad S^y(1) = S_1^y, \quad S^z(1) = S_1^z. \tag{8}$$

The parameter V_1 denotes the velocity of envelope motion of the magnetic soliton. The real constants z_1 and ϕ_1 represent the center position and the initial phase, respectively. The parameter $\zeta_1=\mu_1+i\nu_1$ is the eigenvalue with μ_1, ν_1 being the real and imaginary parts. The one-soliton solution (8) describes a spin precession characterized by four real parameters: velocity V_1, phase Φ_1, the center coordinate of the solitary wave z_1, and initial phase ϕ_1. From the one-soliton solution we obtain the properties of the soliton: (i) both the amplitude and the size of the soliton vary with time periodically, as shown in Fig. 1, in which we demonstrate graphically the dynamics of soliton with the parameters chosen as $\mu_1=0.45$, $\nu_1=0.7$, $\rho^2=0.375$, $z_1=-14$, and $\phi_1=0.5$. This property results from the term ρ in Eq. (5) which is determined by the light-induced dipole-dipole interaction. This

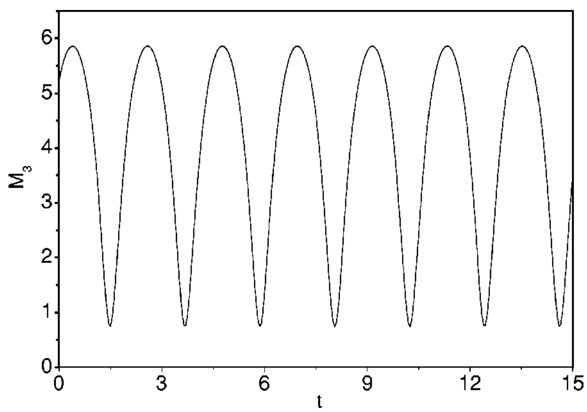

FIG. 2. The magnetization M_3 [the integral $\int_{-\infty}^{\infty} dz(1-S_1^x)$] varies with time periodically, where $\mu_1=0.45$, $\nu_1=0.7$, $z_1=-14$, $\phi_1=0.5$, and $\rho^2=0.375$. The unit for time t is $2\lambda/\hbar$ and a for space z.

significant observation from the one-soliton solution shows that the time oscillation of the amplitude and the size of soliton can be controlled in practical experiment by adjusting of the light-induced dipole-dipole interaction. (ii) The magnetization defined by $M_3=\int_{-\infty}^{\infty} dz(1-S_1^x)$ varies with time periodically as shown in Fig. 2. These properties are similar to that of the Heisenberg spin chain with an easy-plane anisotropy where the dipolar coupling is typically several orders of magnitude weaker than the exchange coupling and thus would correspond to Curie temperatures much below the observed values. Hence the contribution of the dipolar coupling to the spin wave can be neglected in practice. However, for the spinor BEC in the optical lattice the exchange interaction is absent and the individual spin magnets are coupled by the magnetic and the light-induced dipole-dipole interactions. Due to the large number of atoms N at each lattice site, these site-to-site interactions, despite the large distance between sites, explain the natural existence of magnetic soliton which agrees with the results in Refs. [7,9].

To see closely the physical significance of one-soliton solution, it is helpful to show the parameter dependence of Euler angles of the magnetization vector which in a spherical coordinate is

$$S_1^x(z,t) = \cos\theta, \quad S_1^y + iS_1^z = \sin\theta \exp(i\varphi). \tag{9}$$

From Eqs. (6) and (8) we find

$$\cos\theta = 1 - \frac{\dfrac{2\nu_1^2}{|\zeta_1|^2} + 2\chi_{3,1}\sin^2\Phi_1}{\cosh^2[F_1^{-1}(z - V_1 t - z_1)] + \chi_{3,1}\sin^2\Phi_1},$$

$$\varphi = \frac{\pi}{2} + \arctan(\eta_1 \tan\Phi_1) + \arctan(\tanh\Theta_1), \tag{10}$$

where $F_1=1/(2\kappa_{4,1})$ and the maximal amplitude $A_M=2(\nu_1^2/|\zeta_1|^2+|\chi_{3,1}|)$. When $|\zeta_1|^2 \gg \rho^2$, the phase φ can be rewritten as

$$\varphi = \frac{\pi}{2} + \phi_1 + k_1 z - \Omega_1 t + \arctan(\tanh\Theta_1), \tag{11}$$

where the wave number $k_1=2\kappa_{3,1}$ and the frequency of magnetization precession, Ω_1, are related by the dispersion law

$$\Omega_1 = k_1(k_1 + 4\rho^2\mu_1/|\zeta_1|^2) - 4\kappa_{2,1}\kappa_{4,1}. \tag{12}$$

We also see that the position of minimum of energy spectrum $\Omega_{1,\min}=0$ is located at $k_{1\ \min}=[\sqrt{(2\rho^2\mu_1)^2+4\kappa_{2,1}\kappa_{4,1}|\zeta_1|^4}-2\rho^2\mu_1]/|\zeta_1|^2$.

If the amplitude A_M approaches zero—namely, $\nu_1\to 0$—the parameter F_1 diverges and Eq. (10) takes the asymptotic form

$$\cos\theta \to 1, \quad \varphi \to \frac{\pi}{2} + \phi_1 + k_1 z - \Omega_1 t, \tag{13}$$

indicating a small linear solution of magnon. In this case the dispersion law reduces to $\Omega_1=k_1(k_1+4\rho^2\mu_1/|\zeta_1|^2)$.

IV. ELASTIC SOLITON COLLISIONS FOR SPINOR BEC's IN AN OPTICAL LATTICE

The magnetic soliton collision in spinor BEC's is an interesting phenomenon in spin dynamics. Here in terms of a Darboux transformation we first of all give the two-soliton solution (for detail see the Appendix) of Eq. (5),

$$S^x(2) = S_1^x S_2^x + R_3 S_1^y + R_5 S_1^z,$$

$$S^y(2) = S_1^x S_2^y + R_4 S_1^y + R_6 S_1^z,$$

$$S^z(2) = S_1^x S_2^z + R_1 S_1^y + R_2 S_1^z, \tag{14}$$

where S_n^x, S_n^y, and $S_n^z(n=1,2)$ are defined in Eq. (6) and $R_j(j=1,2,\ldots,6)$ take form as follows:

$$R_1 = \frac{1}{\Lambda_2}(\chi_{1,2}\cosh\Theta_2\sinh\Theta_2 - \chi_{2,2}\eta_2\cos\Phi_2\sin\Phi_2),$$

$$R_2 = 1 - \frac{\chi_{2,2}}{\Lambda_2}(\cosh^2\Theta_2 - \sin^2\Phi_2),$$

$$R_3 = \frac{1}{\Lambda_2}(\chi_{1,2}\eta_2\cosh\Theta_2\sin\Phi_2 - \chi_{2,2}\sinh\Theta_2\cos\Phi_2),$$

$$R_4 = 1 - \frac{1}{\Lambda_2}[\chi_{2,2}\sinh^2\Theta_2 + (\chi_{2,2} + 2\chi_{3,2})\sin^2\Phi_2],$$

$$R_5 = \frac{1}{\Lambda_2}(\chi_{2,2}\eta_2\sin\Phi_2\sinh\Theta_2 - \chi_{1,2}\cosh\Theta_2\cos\Phi_2),$$

$$R_6 = \frac{-1}{\Lambda_2}(\chi_{2,2}\eta_2\sin\Phi_2\cos\Phi_2 + \chi_{1,2}\cosh\Theta_2\sinh\Theta_2), \tag{15}$$

where Θ_2, Φ_2, Ω_2, Λ_2, η_2, and $\chi_{m,2}(m=1,2,3)$ are defined in Eq. (7). The solution (14) describes a general elastic scattering process of two solitary waves with different center velocities V_1 and V_2 and different phases Φ_1 and Φ_2. Before collision, they move towards each other, one with velocity V_1 and shape variation frequency Ω_1 and the other with V_2 and Ω_2. In order to understand the nature of two-soliton interactions, we analyze the asymptotic behavior of the two-soliton solution (14). Asymptotically, the two-soliton waves (14) can be written as a combination of two one-soliton waves (8) with different amplitudes and phases. The asymptotic form of two-soliton solution in limits $t\to-\infty$ and $t\to\infty$ is similar to that of the one-soliton solution (8). In order to analyze the asymptotic behavior of two-soliton solutions (14) we show first of all the asymptotic behavior of S_n^x, S_n^y, $S_n^z(n=1,2)$, and $R_j(j=1,2,\ldots,6)$ in the corresponding limits $t\to\pm\infty$ from Eqs. (6) and (15):

$$R_1\to\pm\chi_{1,2},\quad R_2\to 1-\chi_{2,2},\quad R_3\to 0,$$

$$R_4\to 1-\chi_{2,2},\quad R_5\to 0,\quad R_6\to\mp\chi_{1,2},$$

$$S_n^x\to 1,\quad S_n^y\to 0,\quad S_n^z\to 0,\quad \text{as } t\to\pm\infty. \tag{16}$$

Without loss of generality, we assume that $\kappa_{4,n}>0(n=1,2)$ and $V_1>V_2$ which corresponds to a head-on collision of the solitons. For the above parametric choice, the variables Θ_n $(n=1,2)$ for the two-soliton behave asymptotically as (i) $\Theta_1\sim 0$, $\Theta_2\sim\pm\infty$, as $t\to\pm\infty$, and (ii) $\Theta_2\sim 0$, $\Theta_1\sim\mp\infty$, as $t\to\pm\infty$. This leads to the following asymptotic forms for the two-soliton solution. (For the other choices of $\kappa_{4,n}$ and V_n, a similar analysis can be performed straightforwardly).

(i) Before collision—namely, the case of the limit $t\to-\infty$.

(a) Soliton 1 $(\Theta_1\sim 0,\Theta_2\to-\infty)$:

$$\begin{pmatrix} S^x(2)\\ S^y(2)\\ S^z(2)\end{pmatrix}\to\begin{pmatrix} S_1^x\\ \sin\theta\cos(\varphi-\phi_\Delta)\\ \sin\theta\sin(\varphi-\phi_\Delta)\end{pmatrix}, \tag{17}$$

where $\phi_\Delta=\arctan[2\mu_2\nu_2/(\mu_2^2-\nu_2^2)]$ and the parameters θ and φ are defined in Eq. (10).

(b) Soliton 2 $(\Theta_2\sim 0,\Theta_1\to\infty)$:

$$\begin{pmatrix} S^x(2)\\ S^y(2)\\ S^z(2)\end{pmatrix}\to\begin{pmatrix} S_2^x\\ S_2^y\\ S_2^z\end{pmatrix}. \tag{18}$$

(ii) After collision—namely, the case of limit $t\to\infty$.

(a) Soliton 1 $(\Theta_1\sim 0,\Theta_2\to\infty)$:

$$\begin{pmatrix} S^x(2)\\ S^y(2)\\ S^z(2)\end{pmatrix}\to\begin{pmatrix} S_1^x\\ \sin\theta\cos(\varphi+\phi_\Delta)\\ \sin\theta\sin(\varphi+\phi_\Delta)\end{pmatrix}. \tag{19}$$

(b) Soliton 2 $(\Theta_2\sim 0,\Theta_1\to-\infty)$:

$$\begin{pmatrix} S^x(2)\\ S^y(2)\\ S^z(2)\end{pmatrix}\to\begin{pmatrix} S_2^x\\ S_2^y\\ S_2^z\end{pmatrix}. \tag{20}$$

Here for the expressions of solitons before and after collision, S_n^x, S_n^y, and $S_n^z(n=1,2)$ are defined in Eq. (6). Analysis reveals that there is no amplitude exchange among three components S^x, S^y, and S^z for soliton 1 and soliton 2 during collision. However, from Eqs. (17) and (19) one can see that there is a phase exchange $2\phi_\Delta$ between two components S^y and S^z for soliton 1 during collision. This elastic collision between two magnetic solitons in the optical lattice is different from that of coupled nonlinear Schrödinger equations [20]. It shows that the information held in each soliton will almost not be disturbed by each other in soliton propagation. These properties may have potential application in future quantum communication. It should be noted that the inelastic collision may appear if the influence of higher-order terms in Eq. (2) is considered.

V. CONCLUSION

The magnetic soliton dynamics of spinor BEC's in an optical lattice is studied in terms of a modified Landau-Lifshitz equation which is derived from the effective Hamiltonian of a pseudospin chain. The soliton solutions are obtained analytically and the elastic collision of two solitons is demonstrated. The significant observation is that time oscillation of the soliton amplitude and size can be controlled by adjusting of the light-induced dipole-dipole interactions.

It should be interesting to discuss how to create the magnetic soliton and how to detect such magnetic soliton in experiment. In the previous work [5] using Landau-Zener rf sweeps at high fields (30 G) a condensate was prepared in the hyperfine state $|f=1,m_f=0\rangle$—i.e., the ground state of the spinor BEC's. Then the atoms of the ground state can be excited to the hyperfine state $|f=1,m_f=\pm 1\rangle$ by laser light experimentally. Therefore the excited state of the spinor BEC's—i.e., the magnetic soliton—can be created. As can be seen from Fig. 1, the spatial-temporal spin variations in the soliton state are significant. This makes it possible to take a direct detection of the magnetic soliton of spinor BEC's. By counting the difference numbers of the population between the spin +1 and −1 Zeeman sublevel, the average of spin component $\langle S^z\rangle$ is measured directly, while transverse components can be measured by use of a short magnetic pulse to rotate the transverse spin component to the longitudinal direction. Any optical or magnetic method which can excite the internal transitions between the atomic Zeeman sublevels can be used for this purpose. In current experiments in optical lattices, the lattice number is in the range of 10–100, and each lattice site can accommodate a few thousand atoms. This leads to a requirement for the frequency measurement precision of about 10−100 kHz. This is achievable with current techniques. We can also see that the detection of the magnetic soliton of the spinor BEC's is different from that of the Heisenberg spin chain.

The magnetic soliton of spinor BEC's in an optical lattice is mainly caused by the magnetic and the light-induced dipole-dipole interactions between different lattice sites. Since these long-range interactions are highly controllable, the spinor BEC's in optical lattice which is an exceedingly clean system can serve as a test ground to study the static and dynamic aspects of soliton excitations.

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APPENDIX

The corresponding Lax equations for the Eq. (5) are written as

$$\partial_z G(z,t) = LG(z,t), \quad \partial_t G(z,t) = MG(z,t), \tag{A1}$$

where

$$L = -i\epsilon S^z\sigma_3 - i\varsigma(S^x\sigma_1 + S^y\sigma_2),$$

$$M = i2\varsigma^2 S^z\sigma_3 + i2\varsigma\epsilon(S^x\sigma_1 + S^y\sigma_2) - i\varsigma(S^y\partial_z S^z - S^z\partial_z S^y)\sigma_1 - i\varsigma(S^z\partial_z S^x - S^x\partial_z S^z)\sigma_2 - i\epsilon(S_1\partial_z S^y - S^y\partial_z S^x)\sigma_3. \tag{A2}$$

Here $\sigma_j(j=1,2,3)$ is Pauli matrix and the parameters ϵ and ς satisfy the relation $\epsilon^2=\varsigma^2+4\rho^2$. Thus Eq. (5) can be recovered from the compatibility condition $\partial_t L-\partial_x M+[L,M]=0$. We introduce an auxiliary parameter q such that

$$\epsilon = 2\rho\frac{q+q^{-1}}{q-q^{-1}}, \quad \varsigma = 2\rho\frac{2}{q-q^{-1}}, \tag{A3}$$

and the complex parameter is defined by $q=(\zeta+\rho)/(\zeta-\rho)$.

It is easily to see that $S_0=(1,0,0)$ is a simplest solution of Eq. (5). Under this condition the corresponding Jost solution of Eq. (A1) can be written as

$$G_0 = U\exp\{-i\varsigma(z-2\epsilon t)\sigma_3\}, \tag{A4}$$

where $U=\frac{1}{2}\{I-i(\sigma_1+\sigma_2+\sigma_3)\}$ with I denoting unit matrix. In the following we construct the Darboux matrix $D_n(q)$ by using the recursion relation

$$G_n(q) = D_n(q)G_{n-1}(q), \quad n = 1,2,3,\ldots, \tag{A5}$$

where $D_n(q)$ has poles. Since

$$\epsilon(-\bar{q}) = \overline{\epsilon(q)}, \quad \varsigma(-\bar{q}) = -\overline{\varsigma(q)}, \quad L(-\bar{q}) = \sigma_1\overline{L(q)}\sigma_1,$$

$$M(-\bar{q}) = \sigma_1\overline{M(q)}\sigma_1, \tag{A6}$$

we then have

$$G_0(-\bar{q}) = -i\sigma_1\overline{G_0(q)}, \quad G_n(-\bar{q}) = -i\sigma_1\overline{G_n(q)},$$

$$D_n(-\bar{q}) = \sigma_1\overline{D_n(q)}\sigma_1, \tag{A7}$$

where the overbar denotes complex conjugate. Suppose that q_n is a simple pole of $D_n(q)$; then, $-\bar{q}_n$ is also a pole of $D_n(q)$. If $D_n(q)$ has only these two simple poles, we have

$$D_n(q) = C_nP_n(q), \tag{A8}$$

$$P_n(q) = I + \frac{q_n-\bar{q}_n}{q-q_n}P_n + \frac{q_n-\bar{q}_n}{q+\bar{q}_n}\tilde{P}_n, \tag{A9}$$

where C_n, P_n, and $\tilde{P}_n$ are 2×2 matrix independent of q and the terms $(q_n-\bar{q}_n)C_nP_n$ and $(q_n-\bar{q}_n)C_n\tilde{P}_n$ are residues at q_n and $\bar{q}_n$, respectively. From Eq (A7), we have

$$C_n = \sigma_1\bar{C}_n\sigma_1, \quad \tilde{P}_n = \sigma_1\bar{P}_n\sigma_1. \tag{A10}$$

From Eqs. (A2) and (A4) we see that

$$L(q) = -L^\dagger(\bar{q}), \quad M(q) = -M^\dagger(\bar{q}), \quad G_0^{-1}(q) = G_0^\dagger(\bar{q}), \tag{A11}$$

and hence we have

$$G_n^{-1}(q) = G_n^\dagger(\bar{q}), \quad D_n^{-1}(q) = D_n^\dagger(\bar{q}) = P_n^\dagger(\bar{q})C_n^\dagger. \tag{A12}$$

Since $D_n(q)D_n^{-1}(q)=D_n(q)D_n^{-1}(\bar{q})=I$, it has no poles. Then we obtain

$$P_nP_n^\dagger(\bar{q}_n) = 0, \; P_n\left(I - P_n^\dagger + \frac{\bar{q}_n-q_n}{2q_n}\tilde{P}_n^\dagger\right) = 0, \tag{A13}$$

which shows the degeneracy of P_n. One can write $P_n=(g_nw_n)^T(\Upsilon_n\xi_n)$ where the superscript T means transpose. Substituting this expression into Eq. (A3) we obtain

$$P_n(q) = \frac{1}{\Delta_n(q-q_n)(q+\bar{q}_n)} \times \begin{pmatrix} \bar{q}_n|\Upsilon_n|^2+q_n|\xi_n|^2 & 0 \\ 0 & q_n|\Upsilon_n|^2+\bar{q}_n|\xi_n|^2 \end{pmatrix} \times \left\{ q^2\begin{pmatrix} q_n|\Upsilon_n|^2+\bar{q}_n|\xi_n|^2 & 0 \\ 0 & \bar{q}_n|\Upsilon_n|^2+q_n|\xi_n|^2 \end{pmatrix} + q(q_n^2-\bar{q}_n^2)\begin{pmatrix} 0 & \bar{\Upsilon}_n\xi_n \\ \bar{\xi}_n\Upsilon_n & 0 \end{pmatrix} - |q_n|^2\begin{pmatrix} q_n|\xi_n|^2+\bar{q}_n|\Upsilon_n|^2 & 0 \\ 0 & q_n|\Upsilon_n|^2+\bar{q}_n|\xi_n|^2 \end{pmatrix} \right\}, \tag{A14}$$

where

$$\Delta_n = |q_n|^2(|\Upsilon_n|^2+|\xi_n|^2)^2 + |\bar{q}_n-q_n|^2|\Upsilon_n|^2|\xi_n|^2. \tag{A15}$$

To determine ξ_n and Υ_n, we substitute Eq. (A5) into Eq. (A1) and take the limit $q\to q_n$ and then obtain

$$\partial_z D_n(q) = L_n(q)D_n(q) - D_n(q)L_{n-1}(q),$$

$$\partial_t D_n(q) = M_n(q)D_n(q) - D_n(q)M_{n-1}(q). \tag{A16}$$

Because of the degeneracy of P_n, the second factor of the right-hand sides of Eq. (A16)—namely, $(\Upsilon_n\xi_n)G_{n-1}(q_n)$—must appear on the left-hand side in its original form and, hence, it is independent of z and t. We simply let

$$(\Upsilon_n\ \xi_n) = (b_n\ 1)G_{n-1}^{-1}(q_n). \tag{A17}$$

Here b_n is a constant. Hence, the Darboux matrices $D_n(q)$ have been determined recursively, except for C_n. By a simple algebraic procedure, it is seen that Δ_n is always nonvanishing regardless of the values z and t. This shows the regularity of P_n and then $P_n(q)$. In the limit as $q\to 1$, from Eq. (A3) we have

$$\epsilon(q), \quad \mathrm{s}(q) \to 2\rho\frac{1}{q-1} + O(1),$$

and then from Eq. (A16) we obtain

$$S(n)\cdot\sigma = D_n(1)[S(n-1)\cdot\sigma]D_n^{\dagger}(1),\ n=1,2,3,\ldots\ . \tag{A18}$$

Considering Eqs. (A11) and Eq. (A12) we get

$$C_nC_n^{\dagger} = I, \tag{A19}$$

which shows that the matrix C_n is diagonal with the help of the Eq. (A10) and

$$(C_n)_{11} = (\overline{C_n})_{22}, \quad |(C_n)_{11}| = 1. \tag{A20}$$

Then we can write $C_n=\exp(i\omega_n\sigma_3/2)$ which is real and characterizes the rotation angle of spin in the xy plane. It is necessary to mention that ω_n may be dependent on z and t. To determine ω_n one must examine the Lax equations carefully. Since $\exp(i\omega_n\sigma_3/2)$ denotes a rotation around the z axis, it does not affect the value of S^z. Substituting Eq. (A8) into Eq. (A16) and taking the limits as $q\to\infty$ and $q\to 0$, respectively, we obtain

$$\partial_z\{C_nP_n(0)\} = L_n(q)\{C_nP_n(0)\} - \{C_nP_n(0)\}L_{n-1}(q),$$

$$\partial_z\{C_n\} = -i2\rho S^z(n)\sigma_3\{C_n\} + \{C_n\}i2\rho S^z(n-1)\sigma_3.$$

Comparing these two equations, we derive

$$C_n = (\Delta_n)^{-1/2}\begin{pmatrix} q_n|\Upsilon_n|^2 + \overline{q_n}|\xi_n|^2 & 0 \\ 0 & \overline{q_n}|\Upsilon_n|^2 + q_n|\xi_n|^2 \end{pmatrix}. \tag{A21}$$

The Eq. (A17) gives

$$\Upsilon_n = f_n + i\bar{f}_n^{-1}, \quad \xi_n = f_n - i\bar{f}_n^{-1}, \tag{A22}$$

where

$$f_n^2 = \exp(-\Theta_n + i\Phi_n).$$

Here the parameters Θ_n and Φ_n are defined in Eq. (7). Setting $n=1$ and substituting the Eqs. (A8), (A14), (A21), and (A22) into Eq. (A18) we can obtain the one-soliton solution (8). Setting $n=2$ and with the similar procedure the expression of the two-soliton solution (14) is derived.

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