Entanglement control in an anisotropic two-qubit Heisenberg *XYZ* model with external magnetic fields

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We study the bipartite entanglement dynamics in an anisotropic two-qubit Heisenberg *XYZ* model under the influence of population relaxation and in the presence of various types of magnetic fields. We find that the maximal value of the concurrence has an intrinsic rigidity for the different external magnetic fields. Moreover, our results demonstrate that it is possible to produce entangled states and to control or to modulate the concurrence within the intrinsic maximal value with the help of external time-varying fields despite the existence of dissipation.

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I. INTRODUCTION

Entanglement, as a nonlocal quantum correlation without any classical analog, has aroused great interest since the early days of quantum mechanics, and become a central issue in the recently emerged field of quantum information due to its potential applications in the quantum cryptography, teleportation, and quantum computing, etc. [1]. However, when it comes to generate, observe or manipulate entanglement in practice, the fragility of quantum entanglement to the environment induced decoherence often constitutes a major obstacle. Therefore, how to generate, maintain, and control the entanglement in the presence of dissipative coupling of the system to the environment is of utmost importance in the implementation of quantum information processing.

Numerous works have been devoted to entanglement characterization, entanglement control, and entanglement production in solid-state systems, which show promising features as far as the crucial scalability property is concerned [2,3]. In recent literature there has been much interest in entanglement in Heisenberg spin chain [4–9]. This is because the Heisenberg spin systems are, on the one hand, natural candidates for realizing entanglement and for simulating the interactions between qubits; on the other hand, they also serve as the models for various solid-state quantum computation schemes. For instance, the Heisenberg chain has been used to construct a quantum computer based on, respectively, quantum dots [10], nuclear spins [3], electronic spins [11], and optical lattices [12]. Dissipative effects on entanglement in Heisenberg chains and other systems have recently been analyzed in several contexts both in the absence and in the presence of an external homogeneous static magnetic field [4,9,13–16].

In the present paper, we will focus on the interplay between dissipation and various external magnetic fields for bipartite entanglement in an anisotropic two-qubit Heisenberg *XYZ* model. The measure for bipartite entanglement we use is the concurrence, while the dissipation we consider is the population relaxation of the higher level in the qubit due to the environment. The cases of external magnetic fields include: an inhomogeneous static field, an exponentially varying field, and two periodically varying fields. The case of two-qubit Heisenberg *XY* model in a homogeneous magnetic field is studied in Ref. [9]. However, there not only exists the possibility of unwanted inhomogeneous Zeeman coupling in reality [17], but there are also some schemes that employ inhomogeneous Zeeman coupling for faster gate operations [18]. Besides these, in NMR quantum computing, a series of magnetic pulses are applied to a selected nucleus of a molecule to implement quantum gates. In these cases, it will be necessary to obtain the information about the entanglement characteristics of the open system under magnetic field pulses. The present study of entanglement in simple examples will help us to understand the dissipative behavior of the bipartite entanglement subject to various external magnetic fields. More importantly, we will demonstrate that one can control or manipulate the entanglement within the intrinsic maximal value in spin system with the help of external magnetic fields.

II. THE MODEL AND DISSIPATIVE DYNAMICS

The Hamiltonian for an anisotropic N-qubit Heisenberg chain with nearest-neighbor interactions can be written as

$$H=\sum_{n=0}^{N}(J_xS_n^xS_{n+1}^x+J_yS_n^yS_{n+1}^y+J_zS_n^zS_{n+1}^z), \tag{1}$$

where $S_n^\alpha=\frac{1}{2}\sigma_n^\alpha(\alpha=x,y,z)$ and σ_n^α are the local spin-$\frac{1}{2}$ operators and Pauli operators, respectively, at site $n,\hbar=1$, and the periodic boundary condition $S_{N+1}=S_1$ applies. J_α's are real coupling constants for the spin interaction.

We consider an anisotropic two-qubit Heisenberg *XYZ* model subject to an external magnetic field. Two-qubit system is not only essential to the construction of the universal gate, but also the most fundamental case on which further extensions can be developed. We first study the case of inhomogeneous static magnetic field. When the external field B is parallel to the z axis, the Hamiltonian H can be written in terms of the spin raising and lowering operators $S^{\pm}=S^x \pm iS^y$, as

$$H = J(S_1^+S_2^- + S_1^-S_2^+) + \Delta(S_1^+S_2^+ + S_1^-S_2^-) + J_z S_1^z S_z^2 + B_1 S_1^z + B_2 S_2^z, \quad (2)$$

where B_1 and B_2 are the local external magnetic fields at each qubit site, $J=(J_x+J_y)/2$, and $\Delta=(J_x-J_y)/2$ is the anisotropy parameter. The eigenvalues and eigenstates are found as $H|\Psi_{\pm}\rangle=(1/2)(J_z \pm u)|\Psi_{\pm}\rangle$ and $H|\Phi_{\pm}\rangle=(1/2)(J_z \pm v)|\Phi_{\pm}\rangle$, with the eigenstates $|\Psi_{\pm}\rangle=(\pm\sqrt{(u\pm\epsilon)/2u},0,0,\sqrt{2}\Delta/\sqrt{u(u\pm\epsilon)})^T$ and $|\Phi_{\pm}\rangle=(0,\pm\sqrt{(v\pm\epsilon)/2v},\sqrt{2}\Delta/\sqrt{v(v\pm\epsilon)},0)^T$, where $u=\sqrt{4\Delta^2+\epsilon^2}$ and $v=\sqrt{4J^2+\eta^2}$, and $\epsilon=B_1+B_2$, $\eta=B_1-B_2$.

For a bipartite system described by the density matrix ρ, the concurrence between the two qubits, which can be proven to be an entanglement monotone, is [19]

$$C = \max\{\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0\}, \quad (3)$$

where $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ are the eigenvalues of the matrix $R=\rho\tilde{\rho}$, in which $\tilde{\rho}$ is the spin flipped matrix given by $\tilde{\rho}=(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, where σ_y is the usual Pauli matrix, and ρ^* denotes the complex conjugation of the matrix ρ in the standard basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$.

The description of the time evolution of an open system is provided by the master equation, which can be written most generally in the Lindblad form with the assumption of weak system-reservoir coupling and Born-Markov approximation [20,21]. For the case under consideration, the Lindblad operator describes the population relaxation of the upper state of each qubit due to their environment, and thus can be written in the form as $\sqrt{\Gamma}\sigma_-$, where Γ is the relaxation rate which is supposed to be the same for the two qubits and $\sigma_- = |1\rangle\langle 0|$ is the spin lowering operator. The Lindblad equation for our case thus reads

$$\frac{d\rho}{dt} = -i[H,\rho] + \Gamma \sum_{j=1,2} \left[S_j^-\rho S_j^{-\dagger} - \tfrac{1}{2}\{S_j^{-\dagger}S_j^-,\rho\}\right], \quad (4)$$

where { } means anticommutator. In the solution of Eq. (4), we discuss three typical cases with initial states: a separable (unentangled) state, $|\psi\rangle=|00\rangle$; the maximally entangled state, $|\phi\rangle=\frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)$; and the equal mixture of the separable state and the Bell state, $|\psi\rangle=\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)$.

III. EFFECTS OF VARIOUS EXTERNAL MAGNETIC FIELDS

First, according to the time evolution of the reduced density matrix described by (4), we apply an inhomogeneous static magnetic field [Eq. (2)] to the two-qubit system and examine the concurrence evolution. For this case, we get the following analytical results in the standard basis when the system is initially prepared in the separable state,

$$\rho_{11}(t) = \frac{2\Delta^2 e^{-\Gamma t}}{u^2+\Gamma^2}[\cosh(\Gamma t) - \cos(ut)],$$

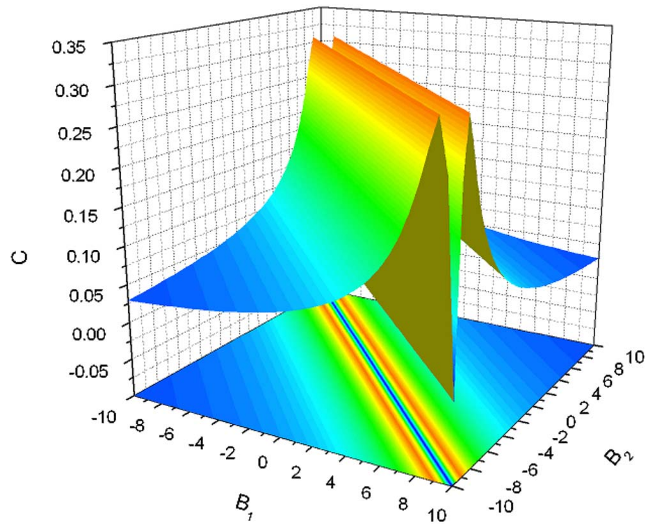

FIG. 1. (Color online) Concurrence vs inhomogeneous static magnetic fields B_1 and B_2 after the evolution time of $t=20$, starting from the separable state. The parameter values for the plot are $J=1.0$, $\Delta=0.3$, $\Gamma=0.3$.

$$\rho_{14}(t) = \frac{\Delta}{u(u^2+\Gamma^2)}[u(\epsilon+i\Gamma)[e^{-\Gamma t}\cos(ut) - 1] + \Delta e^{-\Gamma t}(\Gamma\epsilon - iu^2)\sin(ut)],$$

$$\rho_{22}(t) = \frac{\Delta^2}{u(u^2+\Gamma^2)}[u(1-e^{-2\Gamma t}) - 2\Gamma e^{-\Gamma t}\sin(ut)],$$

$$\rho_{44}(t) = 1 - \rho_{11}(t) - 2\rho_{22}(t). \quad (5)$$

And $\rho_{41}(t)=\rho_{14}^*(t)$, $\rho_{33}(t)=\rho_{22}(t)$, all the other matrix elements are zero. It can be seen that all of the nonzero matrix elements include terms that oscillate with the frequency u and the terms decay at the rate Γ. In addition, if the anisotropy parameter Δ is zero, then the system will just stay unentangled.

Figure 1 shows two concurrence ridges consist of the steady state (maximal concurrence) points which are obtained after the evolution time $t=20$. The parameter values for the plot are $\Delta=0.3$, $\Gamma=0.3$. (Same parameter values are used throughout the paper.) The J is set to 1.0 and all units are scaled by J. Note that we are working in units so that the parameters are dimensionless.

According to Eq. (3), we find the analytic expression for the concurrence ridges as

$$C^{\text{ridge}} = \frac{2\Delta}{u^2+\Gamma^2}(\sqrt{\epsilon^2+\Gamma^2} - \Delta). \quad (6)$$

Very desirable result can be found from Eq. (6) that bipartite entanglement can be produced from the initially separable state despite the presence of decoherence effects as long as the anisotropy parameter is nonzero and $\Delta \neq \sqrt{\epsilon^2+\Gamma^2}$ in the Heisenberg spin chain. Moreover, $C^{\text{ridge}} \to 0$, when (i) $\epsilon \to 0$ or $B_1 \approx -B_2$; (ii) $\epsilon \to \infty$ or $|B_1|,|B_2| \to \infty$, for the fixed Δ and Γ. If we set $d=\Delta/\Gamma$ and $b_{1,2}=B_{1,2}/\Gamma$, the above equation

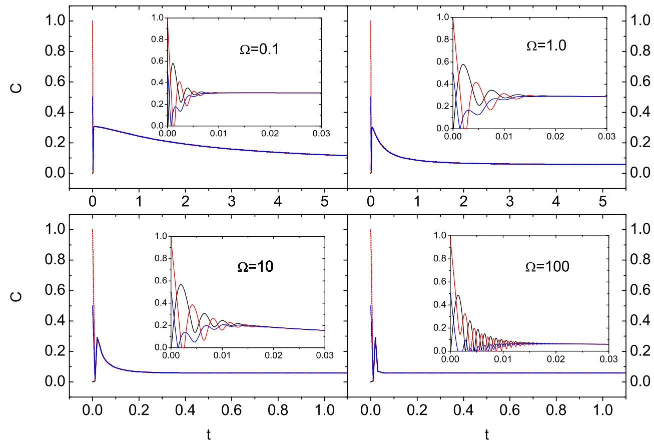

FIG. 2. (Color online) Concurrence evolution of the initially separable states (black), maximum entangled state (red), and equally mixed state (blue) in the presence of the external magnetic field $h_1(t)=b+(a-b)\exp(-\Omega t)$ with different Ω's. The parameter values for the plot are $J=1.0$, $\Delta=0.3$, $\Gamma=0.3$, $a=0.5$, $b=5.0$.

becomes $C^{\text{ridge}}=\frac{2d}{1+4d^2+(b_1+b_2)^2}[\sqrt{1+(b_1+b_2)^2}]$. The two ridges, which are symmetric because the system-environment couplings are the same for the two qubits, give us the corresponding magnetic field values $B_i^{\text{ridge}}=-B_j+\sqrt{-\Gamma^2+(6+2\sqrt{5})\Delta^2}$ (i, $j=1$, 2 and $i\neq j$). Setting this into (6) gives us $C^{\text{ridge}}=0.309$. Another important feature we observed is that, for the other two types of initial entanglement, the plots for final concurrence vs external magnetic fields are almost identical with that in Fig. 1 (omitted). The combinatory effects of the external field and decoherence forces the various initial states to evolve into the final identical one. We also note that, when our case for the external field reduces to the case of the homogeneous one, our results cover those in Ref. [9], which explains that the J_z coupling does not involve the entanglement manipulation by the interplay between the external fields and decoherence in this case.

To describe the concurrence behavior under the fluctuations of external magnetic fields, we also investigate the cases with different homogenous time-dependent, step-function magnetic fields: an exponential and two types of periodic magnetic fields as

$$h_1(t)=\begin{cases} a, & t\leqslant 0,\\ b+(a-b)e^{-\Omega t}, & t>0,\end{cases} \quad (7)$$

$$h_2(t)=\begin{cases} a, & t\leqslant 0,\\ a[1-\sin(\Omega t)], & t>0,\end{cases} \quad (8)$$

$$h_3(t)=\begin{cases} 0, & t\leqslant 0,\\ a[1-\cos(\Omega t)], & t>0,\end{cases} \quad (9)$$

where a,b, and Ω are varying parameters. The fields suffer a sudden change at $t=0^+$.

In Fig. 2, the concurrence evolution subject to a homogeneous exponential magnetic field $h_1(t)$ is shown with different values of Ω, which plays the role of a *knob* adjusting the speed of the passage. We have found out that the concurrence finally reaches a small but steady state value of 0.058 for the parameters chosen regardless of the value of Ω. The insets in Fig. 2 show the evolution of the different initial states in a very short period of time. From these insets, one can see that the combined effect of the external magnetic field and the decoherence forces the various initial entanglement states to oscillate into an identical state, which is eventually lead to the steady state. Especially, for the case of slow passage, e.g., $\Omega=0.1$, the inset shows that the identical state entanglement reaches to the maximal value of 0.309 in a very short period of evolution time, which is the same case with that in Ref. [9]. This is straightforward since the Ωt is very small in this case, so the external field is almost constant thus giving the same result as the case with the external homogeneous magnetic field. As Ω is increased (faster passage), the external magnetic field suffers a sharper increase resulting in the more oscillations of the various initial concurrence and faster evolution of the identical state into the steady state. We also studied the case when the external magnetic field decays exponentially. In this case, the system reaches to the final steady state with the maximal value 0.309 regardless of the

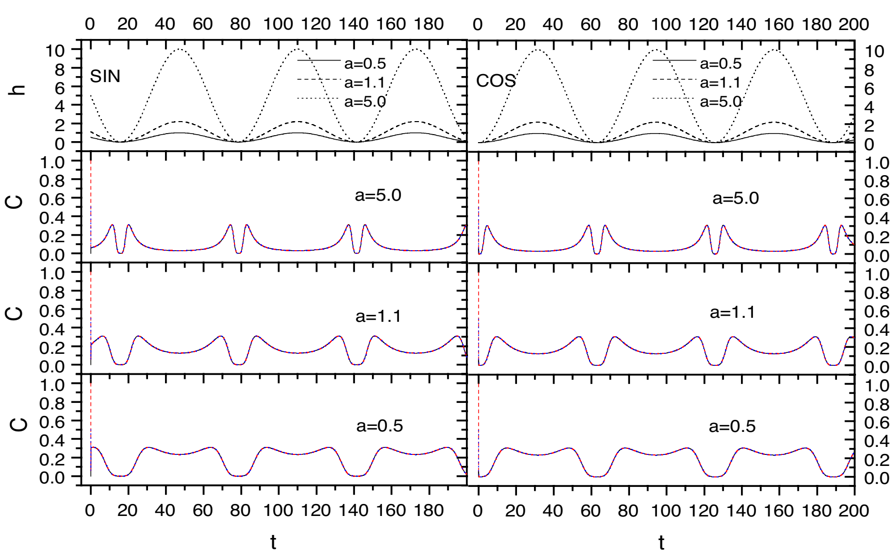

FIG. 3. (Color online) Concurrence evolution of the initially separable states (black), maximum entangled state (red), and equally mixed state (blue) in the presence of the external magnetic field $h_2(t)=a[1-\sin(\Omega t)]$ (left-hand column) and $h_3(t)=a[1-\cos(\Omega t)]$ (right-hand column) with different initial values a of the external magnetic field. The parameter values for the plot are $J=1.0$, $\Omega=0.1$, $\Delta=0.3$, $\Gamma=0.3$.

initial states, which manifests again the intrinsic maximal rigidity of concurrence to the external field parameters, but with different speeds according to the *knob* Ω (plots omitted).

As the next step, we examine the dissipative evolution of concurrence in homogeneous periodic magnetic fields $h_2(t)$ and $h_3(t)$, respectively. The periodic external magnetic fields varying with time are shown in the top panels in Fig. 3 with different initial values a in order for comparison with the concurrence evolution. The parameter values are $b=5.0$ and $\Omega=0.1$. Figure 3 depicts a phenomena that, for all a, three different initial states under the combined effects of the external magnetic fields and the decoherence, instantaneously evolve into the identical state (after some oscillations which cannot be visible in this figure due to its rapidness), which is the same case to the former ones. Interestingly, the evolution patterns of the identical state show the same periodic structure with the external field. The concurrence decreases to the minimal value (zero) due to the decoherence when the external magnetic field reaches its minimum (zero). However, it is not the maximal external field which leads the concurrence to the maximal value. Maximal concurrence 0.309 occurs just around the minimum of the external field, whenever the field is reaching or leaving its minimum. The larger the initial field is, the sooner the concurrence reaches its maximal value. Again, the entanglement-suppressing tendency of the external magnetic field for this system can easily be seen by comparing the plots with different initial magnetic fields. As the initial field is decreased, the curves between the concurrence peaks begin to flatten out. All of the above are the common characteristics of concurrence evolution for both of the periodic fields. The only difference between the two cases is that, when there is no background external field as the case of $h_3(t)$, the identical state entanglement in the very beginning of the evolution is zero since, just at the moment when the external field is turned on, the external field is too small in the competition with the decoherence to produce a nonzero entanglement. However, the concurrence exhibits a revival soon after the external field reaches a competent value.

IV. CONCLUSIONS

We have presented a detailed investigation on the dissipative features of entanglement in an anisotropic two-qubit Heisenberg *XYZ* model subject to various types of external magnetic fields: an inhomogeneous static field; homogeneous exponentially varying and periodically varying magnetic fields, respectively. As shown in the Introduction, the first case is not only pertinent to more realistic models, but also useful in some quantum computing schemes, while the latter two maybe employed in maintaining and manipulating the entanglement in some desired ways. More specifically, when the system is initially prepared in the separable state, controlled generation of bipartite entanglement is possible for all of the different external fields. For the initially separable state, we have given the analytic solutions for the master equation in the standard basis; and also presented a detailed numerical analysis for the other two types of initial states. Furthermore, our results reveal that, by applying an exponentially varying magnetic field, the final entanglement in this system can be maintained at the value within the intrinsic maximal concurrence; one can also modulate the entanglement periodically with the help of periodic magnetic fields, which might be useful in NMR quantum computing. All of the above results not only provide us with useful information about the dissipative dynamics of entanglement in Heisenberg models subject to various magnetic fields, but also might be helpful in the investigation of entanglement production and manipulation in the spin system.

The decoherence scenario considered in this work is the simplest but physically fundamental and unavoidable one. Nevertheless, it would be of more practical importance to extend this case to the more realistic decoherence scenarios and more qubits, such as non-zero temperature bath and pure dephasing since, in reality, there are definitely many degrees of freedom in the system-environment coupling, especially in the solid-state quantum information processing systems.

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