Formation and transformation of vector solitons in two-species Bose-Einstein condensates with a tunable interaction

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Under a unified theory we investigate the formation of various types of vector solitons in two-species Bose-Einstein condensates with arbitrary scattering lengths. We then show that by tuning the interaction parameter via Feshbach resonance, transformation between different types of vector solitons is possible. Our results open up alternate ways in the quantum control of multispecies Bose-Einstein condensates.

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Since the first realization of Bose-Einstein condensate (BEC), a tremendous amount of research has taken place in this interdisciplinary field [1,2]. One specific topic of wide interest is multicomponent BECs, which possess complicated quantum phases [3] and properties not seen in individual components. Theoretical [4–6] and experimental [7–9] studies have shown that interspecies interaction plays crucial roles in these systems. Recently, two-species BECs with tunable interaction [10,11] were successfully produced. This important progress motivates further explorations of two-species BECs.

Solitons represent a highly nonlinear wave phenomenon with unique propagation features and have attracted great interests from many different fields, e.g., nonlinear optics [12]. Because the nonlinear Schrödinger equation governing matter-wave solitons [2,13–15] is similar to those for solitons in other contexts, well-established understandings of solitons can be applied to atomic BECs to achieve better control of matter waves. However, BEC systems possess many unique features of their own and make it possible to study soliton phenomena under circumstances that cannot be easily realized in other fields.

In this paper, we provide a unified theory to investigate various types of vector solitons in two-species condensates with arbitrary scattering lengths. Remarkably, we find that by tuning the interspecies interaction via the well-established Feshbach resonance technique [10], different types of vector solitons can be realized and even transformed to each other. In addition, these vector solitons can be dynamically stable in the presence of soft trapping potentials. Our theoretical work opens up alternative possibilities in the quantum control of multispecies BECs and other related complex systems.

The mean-field dynamics of a two-species BEC is governed by the following equations [2,4–6]:

$$i\hbar\frac{\partial\psi_1}{\partial t}=\left(-\frac{\hbar^2\nabla^2}{2m_1}+V_1+U_{11}|\psi_1|^2+U_{12}|\psi_2|^2\right)\psi_1,$$

$$i\hbar\frac{\partial\psi_2}{\partial t}=\left(-\frac{\hbar^2\nabla^2}{2m_2}+V_2+U_{21}|\psi_1|^2+U_{22}|\psi_2|^2\right)\psi_2,$$

where the condensate wave functions are normalized by particle numbers $N_i\equiv\int|\psi_i|^2d^3\mathbf{r}$, $U_{ii}=4\pi\hbar^2a_{ii}/m_i$ and $U_{12}=U_{21}=2\pi\hbar^2a_{12}/m$ represent intraspecies and interspecies interaction strengths, respectively, with a_{ij} being the corresponding scattering lengths and m the reduced mass. The trapping potentials are assumed to be $V_i=m_i[\omega_{ix}^2x^2+\omega_{i\perp}^2(y^2+z^2)]/2$. Further assuming $\omega_{i\perp}\gg\omega_{ix}$ such that the transverse motion of the condensates are frozen to the ground state of the transverse harmonic trapping potential, the system becomes a quasi-one-dimensional system. Integrating out the transverse coordinates, the resulting equations for the axial wave functions $\tilde{\psi}_{1,2}(x)$ in dimensionless form can be written as

$$i\frac{\partial\tilde{\psi}_1}{\partial t}=\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{\lambda_1^2}{2}x^2+b_{11}|\tilde{\psi}_1|^2+b_{12}|\tilde{\psi}_2|^2\right)\tilde{\psi}_1, \quad (1)$$

$$i\frac{\partial\tilde{\psi}_2}{\partial t}=\left(-\frac{\kappa}{2}\frac{\partial^2}{\partial x^2}+\frac{\lambda_2^2}{2\kappa}x^2+b_{21}|\tilde{\psi}_1|^2+b_{22}|\tilde{\psi}_2|^2\right)\tilde{\psi}_2, \quad (2)$$

where we have chosen $\sqrt{\hbar/(m_1\omega_{1\perp})}$ and $2\pi/\omega_{1\perp}$ to be the units for length and time, respectively; and $\tilde{\psi}_{1,2}$ is normalized such that $\int|\tilde{\psi}_1|^2dx=1$ and $\int|\tilde{\psi}_2|^2dx=N_2/N_1$. Other parameters in Eqs. (1) and (2) are defined as $b_{11}=2a_{11}N_1$, $b_{12}=b_{21}=2m_1a_{12}N_1/[(1+\gamma)m]$, $b_{22}=2a_{22}m_1N_1\gamma/m_2$, $\gamma=\omega_{2\perp}/\omega_{1\perp}$, $\lambda_1=\omega_{1x}/\omega_{1\perp}$, $\lambda_2=\omega_{2x}/\omega_{1\perp}$, and $\kappa=m_1/m_2$.

A unified treatment of different vector-soliton solutions. When the longitudinal trapping potential is neglected ($\lambda_i=0$), Eqs. (1) and (2) are perfectly integrable under the condition $\kappa=1$ (i.e., $m_1=m_2$) and $b_{11}=b_{12}=b_{21}=b_{22}$ [16], allowing for a general procedure to construct two-component vector-soliton solutions in the form of "dark-dark" [17], "bright-dark" [18], and "bright-bright" solitons [19]. When integrability is destroyed, previous studies show that distorted versions of soliton solutions exist, but closed form can only be given for special cases and often for one particular type of vector soliton [20]. Here we show that, even when the above-mentioned integrability condition is violated (e.g., for arbitrary interaction strengths b_{ij} and mass ratio κ), there

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still exists in general a specific class of exact solutions that include all types of vector solitons, so long as $b_{12}^2 \neq b_{11}b_{22}$. The type of vector soliton is determined by the values of b_{ij} and κ.

The vector-soliton solution can be found by inserting an appropriate ansatz (see below) into Eqs. (1) and (2). Defining the following two quantities:

$$C_1 \equiv (b_{22} - \kappa b_{12})/(b_{12}^2 - b_{11}b_{22}),$$

$$C_2 \equiv (b_{12} - \kappa b_{11})/(b_{12}^2 - b_{11}b_{22}),$$

the conditions under which various vector-soliton solutions exist are found to be

$$C_1 > 0,\ C_2 < 0\text{: bright-bright (BB)},$$

$$C_1 > 0,\ C_2 > 0\text{: bright-dark (BD)},$$

$$C_1 < 0,\ C_2 > 0\text{: dark-dark (DD)},$$

$$C_1 < 0,\ C_2 < 0\text{: dark-bright (DB)}.$$

Explicit expressions for these vector solitons are then found from the ansatz we use. In particular, the BB solution (i.e., bright soliton for species 1 and bright soliton for species 2; analogous convention applies to all other solutions) is given by

$$\tilde{\psi}_{1B} = \eta\sqrt{C_1}\,\text{sech}(\eta x - \eta v t)e^{i[vx+(\eta^2-v^2)t/2]},$$

$$\tilde{\psi}_{2B} = \eta\sqrt{-C_2}\,\text{sech}(\eta x - \eta v t)e^{i[vx/\kappa+(\kappa\eta^2-v^2/\kappa)t/2]}.$$

The BD solution is given by

$$\tilde{\psi}_{1B} = \eta\sqrt{C_1}\,\text{sech}(\eta x - \eta v t)e^{i[vx+f_1 t]}, \tag{3a}$$

$$\tilde{\psi}_{2D} = [iv\sqrt{C_2}/\kappa + \eta\sqrt{C_2}\tanh(\eta x - \eta v t)]e^{if_2 t}, \tag{3b}$$

with $f_1 = (\eta^2 - v^2)/2 - b_{12}C_2(\eta^2 + v^2/\kappa^2)$ and $f_2 = -b_{22}C_2(\eta^2 + v^2/\kappa^2)$. The DD solution is found to be

$$\tilde{\psi}_{1D} = [iv\sqrt{-C_1} + \eta\sqrt{-C_1}\tanh(\eta x - \eta v t)]e^{if_1 t},$$

$$\tilde{\psi}_{2D} = [iv\sqrt{C_2}/\kappa + \eta\sqrt{C_2}\tanh(\eta x - \eta v t)]e^{if_2 t},$$

with $f_1 = -\eta^2 - v^2(-b_{11}C_1 + b_{12}C_2/\mu^2)$ and $f_2 = -\mu\eta^2 - v^2(-b_{21}C_1 + b_{22}C_2/\mu^2)$. Finally, the DB solution can be obtained from the BD solution (3) by exchanging indices 1 and 2, and let $C_{1,2} \to -C_{1,2}$. In all these cases, the parameter η determines the width of the soliton and can be found by the normalization condition for $\tilde{\psi}_{1,2}$. The parameter v gives the velocity of the soliton. We note that the BB solution under the condition $b_{11}, b_{22} > 0$ and $b_{12} < 0$, but not the other cases, has been studied by Pérez-García and Beitia [21]. The exact expressions for vector solitons found here bear an apparent resemblance to those for integrable systems [16].

Dynamics and dynamical stability. To be specific, we take ^{7}Li as species 1 and ^{23}Na as species 2, with fixed and realistic intraspecies interaction strength $b_{11} < 0$, $b_{22} > 0$ and variable interspecies interaction strength b_{12}. Figure 1 depicts the "phase diagram" for different regimes of vector solitons as b_{12} is varied. As is indicated in Fig. 1, the system supports BB ($b_{12} < \kappa b_{11}$), BD ($\kappa b_{11} < b_{12} < b_{22}/\kappa$), and DD ($b_{12} > b_{22}/\kappa$) vector soliton, while the condition for DB soliton can never be satisfied for this particular ^{7}Li-^{23}Na system.

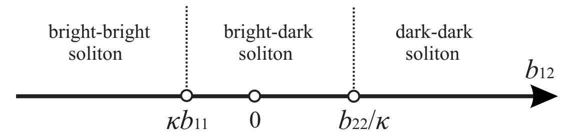

FIG. 1. Phase diagram of vector solitons vs interspecies interaction b_{12} for the system with $b_{11} < 0$ and $b_{22} > 0$. As described in the text, the unit of b_{ij} is $\sqrt{\hbar/(m_1\omega_{1\perp})}$. In the examples shown later in the paper, the two species experience the same trapping frequency, $\omega_{1\perp} = \omega_{2\perp} = 2\pi \times 710$ Hz. The number of atoms in the first species (^{7}Li) is assumed to be 5000, which results in the dimensionless intraspecies interaction strength $b_{11} = -1.47$ and $b_{22} = 5.89$. The two critical values of the interspecies interaction strength are therefore $\kappa b_{11} = -0.45$ and $b_{22}/\kappa = 19.35$.

To study the dynamical stability of the vector-soliton solutions given above, we first calculate the collective excitation frequencies by solving the corresponding Bogoliubov–de Gennes equation [6]. Our calculations show that the excitation frequencies of the BB and BD vector solitons are all real, while those of the DD vector soliton contain complex values. We hence conclude that, for the ^{7}Li-^{23}Na system, the BB and BD vector solitons are dynamically stable, while the DD vector soliton is dynamically unstable.

This stability analysis is confirmed by direct numerical simulations of Eqs. (1) and (2). Computational examples are shown in Fig. 2. In our simulation, a harmonic trapping potential is added along the longitudinal direction to represent a more realistic situation. The trapping potential is weak such that its variation across the soliton scale is negligible. Such a soft trapping potential spatially confines the solitons without affecting their essential properties. The upper panel of Fig. 2 shows the evolution of a BD vector soliton. For the initial condition, we use the exact solution of Eq. (3a) for the bright component, while in order to confine the dark soliton inside the trap, we multiply Eq. (3b) by a Thomas-Fermi profile with an inverted parabolic shape to simulate the dark component which is a quite standard practice [18]. The presence of the weak longitudinal trap causes the vector soliton to oscillate while maintaining the overall shape as shown in the upper panel of Fig. 2. By contrast, the lower panel of Fig. 2 shows that the DD soliton is dynamically unstable—an initial DD vector soliton is quickly destroyed and the two species tend to phase separate into different spatial regions due to the large interspecies repulsion. We remark that there exist alternative ways to study the soliton dynamics, for example, the variational approach used in Ref. [22].

Soliton transformation via Feshbach resonance. Controlling the interaction strength via Feshbach resonace is a unique possibility afforded by atomic systems. Figure 3 shows an example of dynamical conversion between different types of vector solitons via an interspecies Feshbach resonance. Initially, we prepare a zero-velocity BD vector

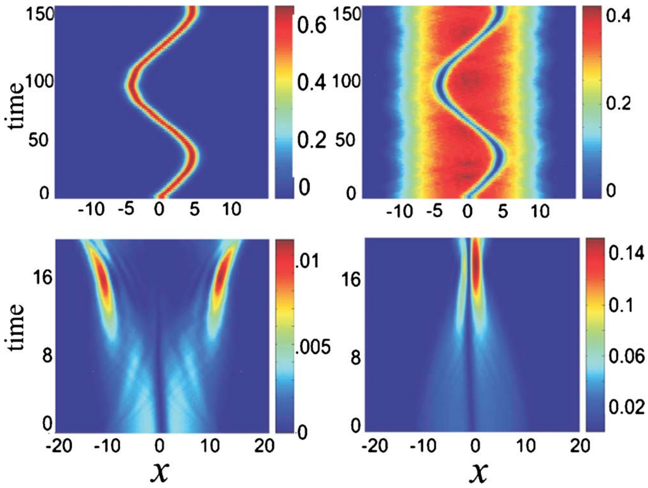

FIG. 2. (Color online) Dynamics of vector solitons inside a weak longitudinal trap with $\lambda_1=\lambda_2=0.1$. Other parameters are specified in Fig. 1. Units for x and time are $\sqrt{\hbar/(m_1\omega_{1\perp})}$ and $2\pi/\omega_{1\perp}$. The left and right plots represent the longitudinal density profiles of the first and second species, respectively. Upper panel: evolution of a BD vector soliton with $b_{12}=2$ initially located at $x=0$ and velocity $v=0.1$. Lower panel: evolution of a DD vector soliton with $b_{12}=30$. For the system under study, the BD (DD) vector soliton is dynamically stable (unstable).

soliton by choosing a proper b_{12} whose value is tuned in later time. In the upper panel of Fig. 3, b_{12} is changed to a value less than κb_{11}. For this value of b_{12}, the system supports BB, but not BD, vector solitons (see Fig. 1). The upper panel of Fig. 3 shows that the initial BD vector soliton is indeed destroyed after b_{12} is changed to the new value. The system first evolves into two pairs of BB vector solitons with finite relative velocity. These two pairs go through several collisions and eventually merge into one single pair. During the evolution, significant radiation from the second species is observed. According to the exact BB solution given above, the number of atoms in each species satisfy

$$\frac{N_1}{N_2}=\frac{C_1}{-C_2}=\frac{b_{22}-\kappa b_{12}}{\kappa b_{11}-b_{12}}.$$

Using realistic values of b_{11} and b_{22} (as detailed in the caption of Fig. 1), we have $N_1/N_2=0.935$. Numerically, we found this ratio to be 0.92–0.95, which is in good agreement with the theoretical value. It is simple to check and intuitive to understand that when the initial BD vector soliton breaks into two pairs of BB solitons, the relative phase of the two bright solitons of the first species is zero, while that of the second species is π. Just like in the scalar condensate case, this leads to attractive (repulsive) soliton-soliton interaction within the first (second) species. We also remark that the relative velocity of the BB vector-soliton pairs thus generated is sensitive to how fast b_{12} is varied. In general, the faster b_{12} is varied, the larger the relative velocity. If we vary b_{12} much slower, the pairs (with smaller relative velocity) will merge together later and experience more collisions. This observation suggests that varying the sweeping rate of b_{12} should help understand more about the collision dynamics of multiple BB vector-soliton pairs [21,23].

In the example shown in the lower panel of Fig. 3, b_{12} is changed to a value larger than b_{22}/κ, i.e., into the DD vector-soliton regime. Here the BD vector soliton does not evolve into a DD vector soliton, which is consistent with the fact that the DD solution is unstable. Instead, due to the strong interspecies repulsion, the core region of the initial dark soliton of species 2 is widened, and is no longer accurately described by a tanh function. In the meantime, the bright soli-

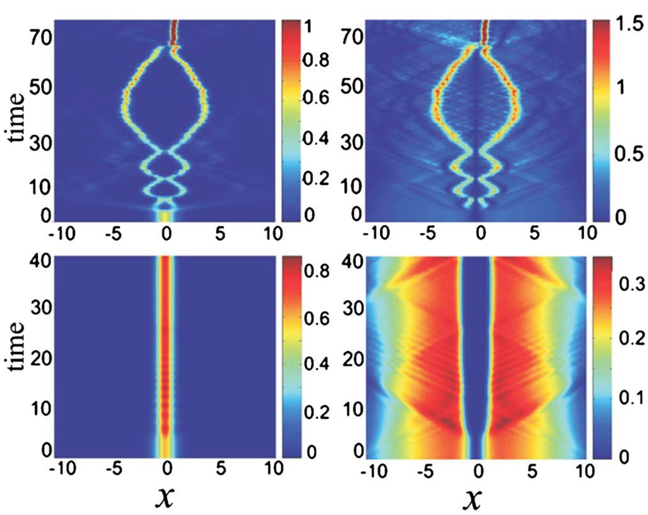

FIG. 3. (Color online) Conversion between different types of vector solitons via Feshbach resonance. The left and right plots represent the longitudinal density profiles of the first and second species, respectively. Units for x and time are $\sqrt{\hbar/(m_1\omega_{1\perp})}$ and $2\pi/\omega_{1\perp}$. Initially, a BD vector soliton is prepared with $b_{12}=2$. Other parameters are the same as in Fig. 2. Upper panel: From $t=3$ to 4, b_{12} is linearly varied to a final value of -10 and stays at that value afterwards. Lower panel: From $t=3$ to 23, b_{12} is linearly varied to a final value of 30 and stays at that value afterwards.

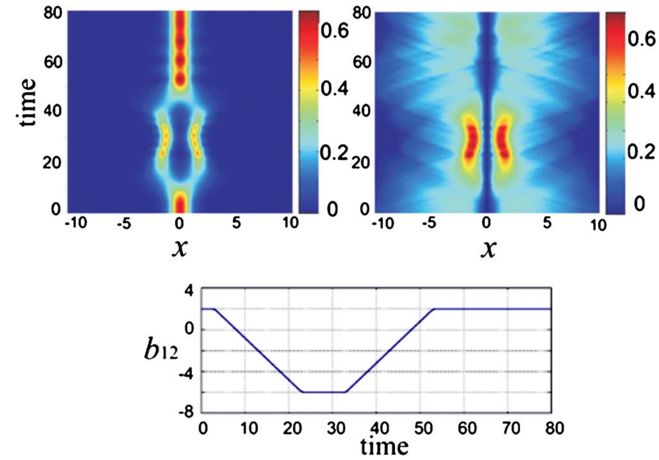

FIG. 4. (Color online) Conversion between BB and BD vector solitons. The left and right plots of the upper panel represent the longitudinal density profiles of the first and second species, respectively. The lower panel shows the variation of b_{12}. Unit for x and b_{12} is $\sqrt{\hbar/(m_1\omega_{1\perp})}$, and unit for time is $2\pi/\omega_{1\perp}$. Initially, a BD vector soliton is prepared with $b_{12}=2$. Other parameters are the same as in Fig. 2.

ton of species 1 is narrowed and remain captivated inside the core of the dark component. This property may help to produce and stabilize bright solitons. During the evolution, we can also see that additional dark solitons can be generated from the core region of species 2 (see the lower right figure in Fig. 3). Of particular interest are those dark solitons generated at early times: They move away from the core, then are bounced back by the trapping potential, and finally are reflected by the central BD vector soliton. Recently similar behavior—the reflection of dark soliton by a BD vector soliton—was observed experimentally [15].

Figure 4 illustrates another example of the conversion between different types of vector solitons via interspecies Feshbach resonance. The initial condition is the same as in Fig. 3 where a zero-velocity BD vector soliton is prepared. The value of b_{12} is then decreased and the system evolves to two BB vector-soliton pairs as in Fig. 3. Then, we restore the initial value of b_{12} before the two BB vector solitons merge into one. As seen in Fig. 4, such a controlled tuning of b_{12} can also recover the initial BD vector soliton. This represents a remarkable example of the coherent control over the BEC dynamics, afforded by the tunability of interaction strength.

In conclusion, we put forward a unified theory to treat all types of vector solitons on equal footing in two-species BECs with arbitrary scattering length. More importantly, we show that different types of vector solitons can be transformed to each other by varying the interaction strength. Besides its fundamental interest in quantum dynamics, our study may also find applications in the quantum control of multispecies BECs. For example, treating one atomic species as the target, its soliton properties can be controlled by the presence of a second species via the tuning of the interaction strength between them. We hope that these results can stimulate investigation of phenomena in nonlinear complex systems in general and further studies of BEC solitons in particular.

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