Dynamics and modulation of ring dark solitons in two-dimensional Bose-Einstein condensates with tunable interaction

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We investigate both analytically and numerically the dynamics and modulation of ring dark solitons in two-dimensional Bose-Einstein condensates with tunable interaction. The analytic solutions for the ring dark soliton are derived by using a transformation method. A shallow ring dark soliton is stable when the ring is slightly distorted, while for large deformation of the ring, vortex pairs appear and demonstrate different dynamical behaviors: the vortex pairs transform into dark lumplike solitons and revert to ring dark solitons periodically. Moreover, our results show that the dynamical evolution of the ring dark soliton can be dramatically affected by the Feshbach resonance, and the lifetime of the ring dark soliton can be greatly extended, which offers a useful method for observing ring dark solitons in future experiments.

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I. INTRODUCTION

Solitons are fundamental excitations of nonlinear media and have attracted great interest in diverse contexts of science and engineering, such as the dynamics of waves in shallow water, transport along DNA and other macromolecules, and fiber optic communications. The realization of Bose-Einstein condensates (BECs) [1] introduces an unparalleled platform for the study of these nonlinear excitations, where both bright [2] and dark solitons [3–7] have been observed. Very recently, quasi-one-dimensional (1D) dark solitons with very long lifetimes have been created in the laboratory and their oscillations and collisions have been observed [8,9]. This offers an unusual opportunity to study dark solitons and the relevant theories.

Dark solitons are robust localized defects in repulsive BECs, which are characterized by a notch in the condensate density and a phase jump across the center [10]. So far most of the work on dark solitons is limited to one-dimensional BECs ([11]; for a review see [12–14]), where they are both stable and easily controlled in experiments. However, in two dimensions, most types of dark solitons are short lived due to dynamical instability arising from the higher dimensionality [4]. For example, in Ref. [15] it is shown that both 2D dark lumplike and dark stripe solitons will decay into vortex pairs under small transverse perturbations. Also these 2D defects are strongly affected by inhomogeneity of the system and suffer from the snaking instability, just as in optical systems ([10,16] and references therein). Therefore their long-time dynamical behaviors are hard to observe in real experiments.

One candidate for observing the long-time behavior of 2D dark solitons is the ring dark soliton (RDS) which was first introduced in the context of optics [17,18] and then studied in BECs [19]. In a 2D nonlinear homogeneous system, it was first predicted that the instability band of a dark stripe soliton can be characterized by a maximum perturbation wave number $Q_{\max}$ [20]; and if the length of the stripe is smaller than the inverse of the wave number, $L<2\pi/Q_{\max}$, then the stripe can be bent into an annulus with the instability being greatly suppressed. This was further confirmed for BECs even in an inhomogeneous trap, where both oscillatory and stationary ring dark solitons can exist [19]. In addition, the symmetry of the ring soliton determines that it is little affected by the inhomogeneity of the BEC system. Moreover, the study indicated that collisions between the RDSs are quasielastic and their shapes will not be distorted [21]. Due to these specific characteristics of the RDS, it has stimulated great interest in observing 2D dark solitons in BECs [22–25]. Recently, Yang *et al.* gave a proposal for creation of RDSs in experiment [26]. Nevertheless, the generation of RDSs and their dynamical behaviors in BECs have not been observed in real experiments yet. This is because the lifetime of deep RDSs is not long enough for experimental observation, and the stability analysis for shallow RDSs has not been explored thoroughly.

In this paper, we first develop a general effective analytical method to derive solutions for the RDS. This is realized by transforming the Gross-Pitaevskii (GP) equation with trap potential to a standard nonlinear Schrödinger (NLS) equation. Then we numerically solve the 2D GP equation and obtain a stability diagram where a stable domain of shallow RDSs can exist. In the unstable region, we find a transformation process between various dark solitons and vortex pairs. Finally, we show that the lifetime of RDSs can be greatly extended by a Feshbach resonance, which is of particular importance for the experimental observation of RDSs.

II. THE MODEL

A BEC trapped in an external potential is described by a macroscopic wave function $\Psi(\mathbf{r},t)$ obeying the GP equation [27], which reads

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t) + g_0(t)|\Psi(\mathbf{r},t)|^2\right)\Psi(\mathbf{r},t),$$

where the wave function is normalized by the particle number $N=\int d\mathbf{r}|\Psi|^2$ and $g_0(t)=4\pi\hbar^2 a(t)/m$ represents the strength of interatomic interaction characterized by the s-wave scattering length $a(t)$, which can be tuned by Feshbach resonance. The trapping potential is assumed to be $V(\mathbf{r},z)=m(\omega_r^2 r^2+\omega_z^2 z^2)/2$, where $r^2=x^2+y^2$, m is the atom mass, and $\omega_{r,z}$ are the confinement frequencies in the radial and axial directions, respectively. Further assuming that $\Omega \equiv \omega_r/\omega_z \ll 1$ such that the motion of atoms in the z direction is essentially frozen in the ground state $[f(z)]$ of the axial harmonic trapping potential, the system can be regarded as quasi-2D. Then we can separate the degrees of freedom of the wave function as $\Psi(\mathbf{r},t)=\psi(x,y,t)f(z)$, obtaining the 2D GP equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi + \frac{m}{2}\omega_r^2 r^2\psi + g_0(t)\,\eta|\psi|^2\psi,$$

where

$$\eta \equiv \frac{\int dz|f(z)|^4}{\int dz|f(z)|^2}.$$

It is convenient to introduce the scales characterizing the trapping potential: the length, time, and wave function are scaled as

$$x=a_h\tilde{x}, \quad t=\frac{\tilde{t}}{\omega_z}, \quad \psi=\frac{\tilde{\psi}}{a_h\sqrt{4\pi a_0\eta}},$$

respectively, with $a_h=\sqrt{\hbar/m\omega_z}$ and a_0 a constant length that we choose to measure the time-dependent s -wave scattering length. Then the 2D GP equation is reduced to a dimensionless form as

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\nabla^2\psi + g(t)|\psi|^2\psi + \frac{1}{2}\Omega^2 r^2\psi, \tag{1}$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2=\partial^2/\partial r^2+(1/r)\partial/\partial r+\partial^2/\partial\theta^2$, $\Omega=\omega_r/\omega_z$, $g(t)=a(t)/a_0$, and the tilde is omitted for simplicity. This is the basic equation we treat analytically and numerically.

III. TRANSFORMATION METHOD AND ANALYTIC SOLUTION

In order to study the dynamics of ring dark solitons, we consider the solution of Eq. (1) with circular symmetry, $\psi(r,t)$. In the case of $g(t)=C$, where C is a nonzero positive constant, the main difficulty in solving Eq. (1) is the existence of the last term, i.e., the trapping potential. Without a trap, i.e., $\Omega=0$, the system is described by the standard NLS equation

$$i\frac{dQ(R,T)}{dT} + \frac{1}{2}\left(\frac{\partial^2 Q(R,T)}{\partial R^2} + \frac{1}{R}\frac{\partial Q(R,T)}{\partial R}\right) - C|Q(R,T)|^2 Q(R,T) = 0. \tag{2}$$

Under the small-amplitude approximation, Eq. (2) has been transformed to the famous cylindrical Korteweg–de Vries (CKdV) equation by using the perturbation method [17,24]. The CKdV equation is known to be the basic nonlinear equation describing cylindrical and spherical pulse solitons in plasmas, electric lattices, and fluids (see, e.g., [28] for a review) and its exact solution has been derived [29]. Therefore the soliton solutions of Eq. (2) are gained.

However, there is no effective method to solve Eq. (1) generally, especially when the last term is time dependent, $V(t)$. Now we develop a method that can transform the general form of Eq. (1) to Eq. (2) by using the transformation

$$\psi(r,t) = Q(R(r,t),T(t))e^{ia(r,t)+c(t)}, \tag{3}$$

where $R(r,t)$, $T(t)$, $a(r,t)$, and $c(t)$ are assumed to be real functions and the transformation parameters read

$$R(r,t) = \alpha(t)r,$$

$$T(t) = \int \alpha^2(t')dt' + C_0,$$

$$c(t) = \frac{1}{2}\ln\frac{\alpha^2(t)}{C},$$

$$a(r,t) = -\frac{1}{2\alpha(t)}\frac{d\alpha(t)}{dt}r^2, \tag{4}$$

where C_0 is a constant. The condition that such transformations exist is

$$\frac{1}{\alpha(t)}\frac{d^2\alpha(t)}{dt^2} - \frac{2}{\alpha(t)^2}\left(\frac{d\alpha(t)}{dt}\right)^2 - \Omega^2 = 0. \tag{5}$$

Under this condition, all solutions of Eq. (2) can be recast into the corresponding solutions of Eq. (1). So we build a bridge between the extensive RDS study in nonlinear optics (homogeneous system) and the trapped BEC system. Furthermore, it is worth noting that the transformation method can be used to solve the general equation

$$i\frac{du(r,t)}{dt} + D(t)\left(\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,t)}{\partial r}\right) + g(t)|u(r,t)|^2 u(r,t) + V(t)r^2 u(r,t) = 0, \tag{6}$$

when $g(t)$ is proportional to $D(t)$. Equation (6) completely describes the dynamics and modulation of both the electric field in optical systems and the macroscopic order parameter in atomic BECs [in this case, $D(t)$ is a constant] in quasi-2D systems with circular symmetry.

In order to get the transformation condition explicitly, we substitute $y(t)=d\alpha/(\alpha\,dt)$ into Eq. (5) and obtain

$$\frac{dy(t)}{dt} = y^2 + \Omega^2. \tag{7}$$

This is a standard Riccati equation. We can solve it not only for Ω=const, but also for various types of $\Omega^2(t)$, such as $a+b\sin(\lambda t)$, $a+b\cosh(t)$, $ae^{\lambda t}$ (a,b,λ are arbitrary constants), and so on. Thus we can study the dynamics of a system with a time-dependent external trap. In particular, when Ω is independent of t, the solution of Eq. (5) is

$$\alpha(t) = C_1 \operatorname{sech}(\Omega t + C_2), \tag{8}$$

where C_1 and C_2 are the integration constants.

Since under the small-amplitude approximation the solutions of Eq. (2) have been given, then by combining the solutions of Eqs. (2) and (8) we get the corresponding solutions for the RDS in a BEC in an external trap potential. They are exact solutions of the system when the depth of the RDS is infinitely small, which helps us to understand the dynamics of RDSs. However, when the RDSs get deeper, the small-amplitude approximation is invalid and we have to appeal to a numerical simulation. In the following sections, we study the dynamics and stability of RDSs numerically.

IV. STABILITY OF THE SHALLOW RING DARK SOLITON

It is known that, starting from the initial configuration with strict circular symmetry, the RDS will oscillate up to a certain time until instabilities develop: a shallow RDS slowly decays into radiation and for a deep one snaking sets in, leading to formation of vortex-antivortex pairs arranged in a robust ring-shaped array (vortex cluster), because of transverse perturbations.[19] But the stability of the RDS to perturbation in the radial direction has not been analyzed. Here we study numerically the stability of shallow RDSs against a small distortion of the ring shape, which is necessary for a practical experiment.

In the optical system, deformed RDSs on top of a constant background, called elliptic dark solitons, have been studied in [30]. As in the homogeneous case, our results show that deformed shallow RDSs in BECs exist and persist over long times for small distortions, and they become more circular when the radius increases. However, because of the inhomogeneous background in BECs, the evolution and instability of deformed RDSs change dramatically and the system displays diverse behaviors, which are illustrated in this section.

We study the stability of RDSs by solving Eq. (1) numerically with the parameters $g(t)=1$, $\Omega=0.028$, and the initial radius of the RDS $R_0=28.9$. Because of the large initial radius, a reasonable and good approximation is

$$\psi(x,y,0) = (1-\Omega^2 r^2/4)[\cos\phi(0)\tanh Z(r_1) + i\sin\phi(0)], \tag{9}$$

where $r=\sqrt{x^2+y^2}$, $Z(r_1)=(r_1-R_0)\cos\phi(0)$, $r_1=\sqrt{(1-e_c^2)x^2+y^2}$, e_c is the eccentricity of the ring, and $\cos\phi(0)$ is proportional to the depth of the input soliton. When $e_c \neq 0$, R_0 represents the length of the semiminor axis of the elliptical configuration. We take Eq. (9) as the initial configuration; the validity of this has been explained in detail and checked in the previous investigations [17,19].

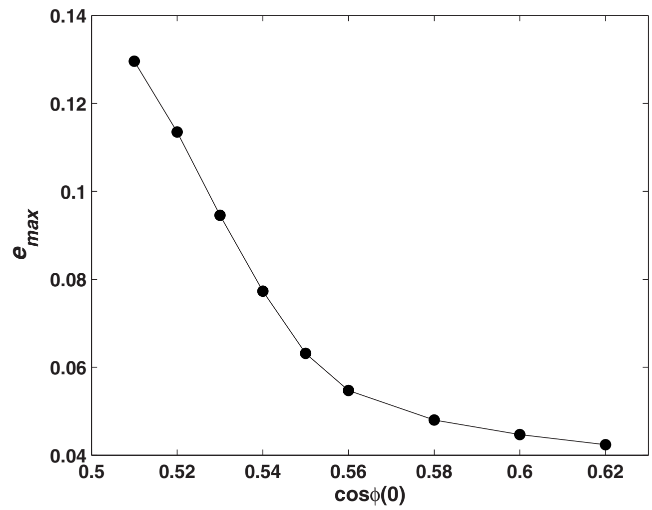

FIG. 1. Part of the stability diagram of a shallow ring dark soliton with $R_0=28.9$ and trap frequency $\Omega=0.028$. The solid line is the instability threshold—below it the deformed ring dark solitons are stable, while above the line snaking sets in. The maximum eccentricity $e_{\max}$ of the ring in the initial configuration decreases as the initial depth $\cos\phi(0)$ of the ring dark soliton increases.

We propagate the 2D time-dependent GP equation [Eq. (1)] using two distinct techniques: the alternating-direction implicit method [31,32] and the time-splitting Fourier spectral method [33]. The results from these two methods are cross checked and when $e_c=0$, the properties of the results agree very well with those in [19].

To translate the results into units relevant to the experiment [3,4], we assume a ^{87}Rb ($a_0=5.7$ nm) condensate of radius 30 μm, containing 20 000 atoms in a disk-shaped trap with $\omega_r=2\pi\times 18$ Hz and $\omega_z=2\pi\times 628$ Hz. In this case, the RDS considered above has the radius $R_0=12.38$ μm, and the unit of time is 0.25 ms.

Our numerical results show that the shallow RDS [this refers to $\cos\phi(0)<0.67$, where the soliton without distortion will not suffer from the snaking instability] is stable against a small distortion of the ring shape. There is a maximum eccentricity $e_{\max}$ for a given initial depth. When $e_c<e_{\max}$, the RDSs are stable and oscillate, preserving their shape with the same period as the unperturbed RDSs until they decay. $e_{\max}$ becomes smaller with increase of the initial depth (see Fig. 1). This is because $2\pi/Q_{\max}$ decreases as the RDS becomes deeper [17]; thus a shallower RDS can support a larger distortion.

When e_c exceeds $e_{\max}$, snaking sets in and the dark soliton breaks into two vortex pairs, presenting a striking contrast to the multiples of four pairs reported previously. The evolution of the vortex pair is very different from the case of deep RDSs without distortion.

To illustrate the generic scenarios, we take a typical case with $\cos\phi(0)=0.6$ and $e_c=0.4$. The RDS initially shrinks and, when the minimum radius is reached in the short-axis direction, it starts snaking and forming two dark lumplike solitons in the horizontal direction; they move in opposite directions and then break into two vortex pairs [see Fig. 2(b)]. The vortex pairs arrange themselves in a ring configu-

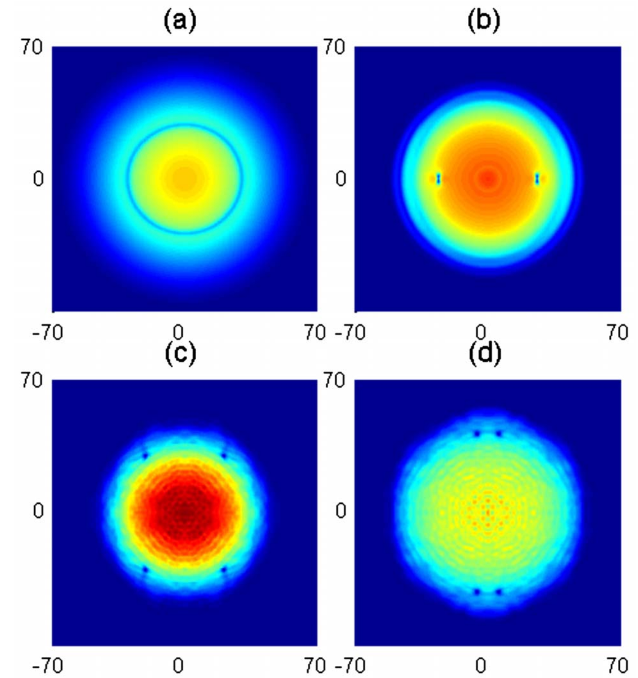

FIG. 2. (Color online) Evolution of the ring dark soliton with initial depth cos $\phi(0)=0.6$, eccentricity $e_c=0.4$, interaction strength $g(t)=1$, and trap frequency $\Omega=0.028$. (a) The initial profile shown by a color-scale density plot, where the ring dark soliton corresponds to the light-color ellipse with the length of the semiminor axis $R_0=28.9$. (b), (c), and (d) correspond to $t=80$, 400, and 540, respectively. The ring dark soliton with distortion first breaks into two lump solitons, and then two vortex pairs. See text for details.

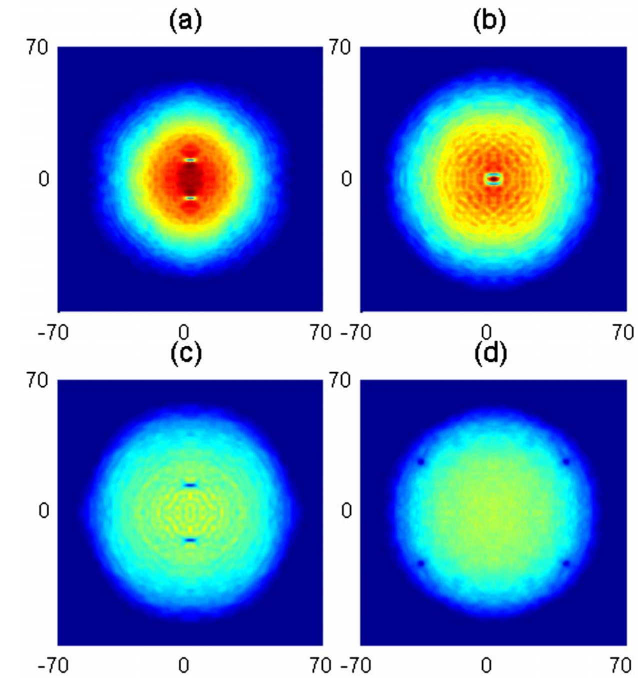

FIG. 3. (Color online) Evolution of vortex pairs following Fig. 2 for $t=$ (a) 640, (b) 660, (c) 680, and (d) 1000. The vortex pairs collide and merge into dark lumplike solitons, which move, then form a ring dark soliton for a short time, and revive. Finally the lumplike solitons again break into vortex pairs. See text for details.

ration, which performs slow radial oscillations. Simultaneously the vortices and antivortices move along the ring [see Figs. 2(c) and 2(d)], The result of the motion is their collision in pairs in the vertical direction, followed by merging of the vortices and antivortices and formation of a pair of lumplike solitons [see Fig. 3(a)]; this process is confirmed by the phase distribution of the system. The lumplike solitons move toward each other, i.e., to the center of the condensate, in the vertical direction. When they reach the minimum radius, a ring dark soliton forms [see Fig. 3(b)]. After a short time the system returns to the state of two dark lumplike solitons. The lumplike solitons leave each other [see Fig. 3(c)] and along the way each lumplike soliton breaks into a vortex pair again. But the configuration is different from the initial one: the vortex (antivortex) is substituted by an antivortex (vortex), just as if the vortex and antivortex passed each other directly, following the motion before merging. Then the vortices and antivortices keep on moving [see Fig. 3(d)], collide, and merge in the horizontal direction. The dynamical process repeats itself.

Part of the process of dynamical behavior of the vortex pairs is similar to the behavior of lump solitons reported in [15]. The certain centrical collision of the vortex pair provides a potential tool to study the collision dynamics of vortices.

V. CONTROL OF DEEP RING DARK SOLITONS

The deep ring dark solitons [with cos $\phi(0)>0.67$] suffer from the snaking instability and can survive for only a short time (typically 10 ms in numerical simulation) before changing into are other soliton type. A longer lifetime is necessary for the practical observation of complex soliton physics such as oscillations or collisions [8]. Our results show that the Feshbach resonance management technique can greatly extend the lifetime of the ring dark solitons, which makes it possible to study the long-time behavior of 2D dark solitons in experiments.

Feshbach resonance [34] is a quite effective mechanism that can be used to manipulate interatomic interaction (i.e., the magnitude and sign of the scattering length), which has been used in many important experimental investigations, such as the formation of bright solitons [2]. Feshbach resonance management refers to the time-periodic changes of the magnitude and/or sign of the scattering length by Feshbach resonance [35], and has stimulated great interest in theoretical [36] and experimental study. Feshbach resonance management has been widely used to modulate and stabilize bright solitons [37,38], while its effect on the dynamics and stability of dark solitons is not very definite [39]. Taking the ring dark soliton as an example, we demonstrate that the Feshbach resonance can dramatically affect the dynamics of dark solitons.

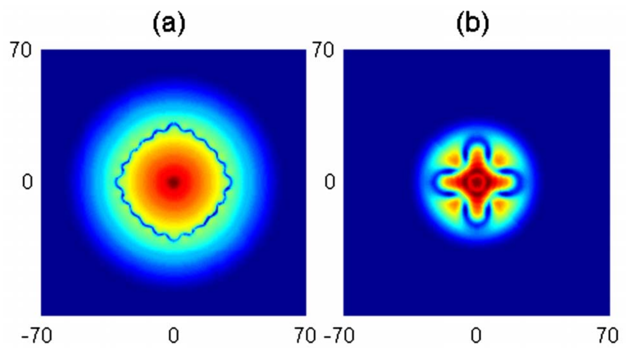

FIG. 4. (Color online) Snapshots of condensate with a ring dark soliton under Feshbach resonance management. The parameters are (a) $e_c=0$, $\cos\phi(0)=1$, $g(t)=2-\sin(2\Omega t)$, and $t=60$ (≈ 15 ms); (b) $e_c=0$, $\cos\phi(0)=1$, $g(t)=e^{-\Omega t}$, and $t=100$ (≈ 25 ms). Other parameters are the same as in Fig. 2. In (a) the ring dark soliton will break into 20 vortex pairs, but only four vortex pairs in (b). The total atom number is the same in these two plots, so for clarity we use different color scales.

To show the effect of Feshbach resonance management (FRM), Eq. (1) is integrated numerically with eccentricity $e_c=0$ and a time-dependent nonlinear coefficient $g(t)$; the other parameters and initial condition are the same as in Sec. IV. We consider the cases of $g(t)\sim ae^{\pm\omega t}$ and $g(t)\sim a\pm\sin(\omega t)$, where a and ω are arbitrary constant parameters. Our results show that, for deep ring dark solitons, the Feshbach resonance management remarkably changes the evolution and the instability of solitons, as indicated by the number of vortex pairs which arise due to the snaking instability. Two typical examples are shown in Fig. 4: in the first, black ring dark solitons [with $\cos\phi(0)=1$] break into 20 vortex pairs, while the second has only four pairs. These phenomena are very different from results with constant scattering length, which have 16 vortex pairs. This makes it possible to study the dynamics of the snaking instability in detail. More interesting and more important are the cases in which the lifetime of the RDS can be extended, such as $g(t)=1-\sin(\omega t)$, as shown in Table I, where the lifetime of RDSs corresponding to different modulation frequencies of the FRM (ω) is given.

TABLE I. The lifetime of a RDS when it suffers the FRM $g(t)=1-\sin(\omega t)$. The parameters are $e_c=0$, $\cos\phi(0)=0.76$. The lifetime refers to the time interval between the start and the time point when snaking sets in.

Frequency of FRM ω (units of Ω)	Lifetime of RDS (ms)
<0.5	<15
0.6	17
0.8	43
1.0	45
1.5	16
>1.7	<15

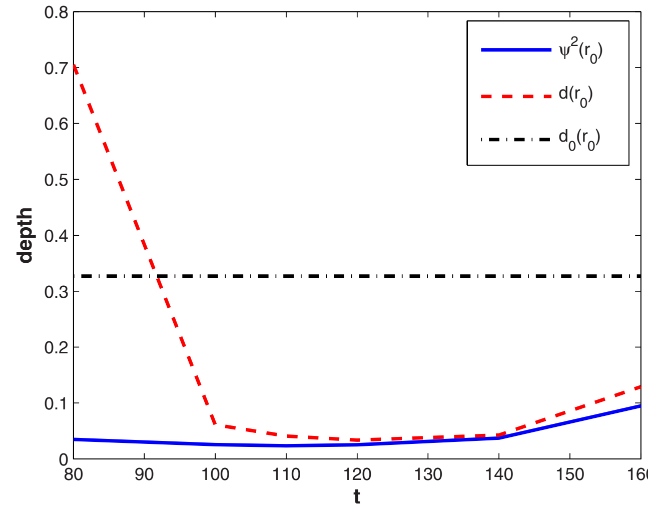

FIG. 5. (Color online) Depth of the ring dark soliton as a function of time. The parameters are $e_c=0$, $\cos\phi(0)=0.76$, and $g(t)=1-\sin(\Omega t)$. $d_0(r_0)$ (black dot-dashed line) is the initial depth of the ring dark soliton, and $d(r_0)$ (red dashed line) is the depth of the ring dark soliton, where r_0 is the position of the center of the ring dark soliton. The depth of the RDS is much smaller than its initial depth at long times due to FRM. $\psi^2(r_0)$ (blue solid line) is the density at the center of the ring dark soliton, which changes very little. This fact shows that the soliton senses a more homogeneous local environment.

In particular, when the system is subject to a modulation of $g(t)=1-\sin(\Omega t)$, the RDS can exist for a longer time before breakup. For example, when $\cos\phi(0)=0.76$, the lifetime can be extended up to 45 ms from 10 ms; in this time, we can observe one complete cycle of oscillation of the RDS. If we just reduce the scattering length but keep it constant, the lifetime of the RDS will not change much. For example, if $g(t)=0.2$, the RDS will start snaking at about 15 ms. So the lifetime extension effect is due to the Feshbach resonance management.

From Eq. (8), we can see that Ω is one intrinsic frequency of the system and the increase of the RDS lifetime is a resonance phenomenon, which is confirmed by Table I. When the modulation frequency is tuned to be resonant with the intrinsic frequency of the system, Ω, on one hand, the RDS is less affected by the inhomogeneous background because of the approximately synchronous movement of the ring dark soliton and the background condensate resulting from FRM; on the other hand, the deep RDSs become effectively shallower [40] (see Fig. 5), prolonging the time needed for the onset of snaking. So the lifetime of the RDS increases.

Furthermore, this effect is even valid for shallow RDSs suffering from the instability due to the distortion of the ring shape. It has been checked that under Feshbach resonance management $g(t)=1-\sin(\Omega t)$, the RDS with $\cos\phi(0)=0.6$ and $e_c=0.4$ has a lifetime of 50 ms, which is much larger than the original 10 ms. Thus under large perturbation the RDS can exist long enough to be observed, which makes the experimental study of RDSs easier.

VI. CONCLUSION

In conclusion, we have introduced a transformation method to derive the solution for the ring dark soliton, which provides a powerful analytic tool to study 2D BECs and nonlinear optical systems with circular symmetry. Then we checked the stability of the shallow ring dark soliton, and found an unusual dynamical behavior mode of solitons. Furthermore, we studied the effect of Feshbach resonance management on the evolution and the stability of ring dark solitons, and discovered a method to greatly extend the lifetime of ring dark solitons. We showed that the ring dark soliton is a potential candidate for observing the long-time behavior of 2D dark solitons.

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