I. INTRODUCTION

In quantum-information theory, entanglement is a vital resource for some practical applications such as quantum cryptography, quantum teleportation, and quantum computation [1,2]. During the last decade, this inspired a great deal of effort for detecting and quantifying the entanglement [3–11]. On the other hand, the creation and distribution of entanglement are also of central interest in quantum-information processing. More especially, the distribution of bipartite entanglement is a key ingredient for performing certain quantum-information processing tasks such as teleportation.

One of the methods, generating bipartite entanglement, is the entanglement of assistance that is defined in Refs. [12,13]. It quantifies the entanglement which could be created by reducing a multipartite entangled state to an entangled state with fewer parties (e.g., bipartite) via measurements. Such producing of entanglement, also called “assisted entanglement,” is a special case of the localizable entanglement [14], which is especially important for quantum communication, where quantum repeaters are needed to establish bipartite entanglement over a long length scale [15]. For a pure $2 \otimes 2 \otimes n$ state, the analytical formula of entanglement of assistance has been derived by Laustsen et al. [16], whereas the calculation of entanglement of assistance is not easy for a general pure tripartite state [17].

In this paper, we explore the entanglement of assistance for a general pure tripartite state in terms of $I$ concurrence [18]. We obtain a lower bound of entanglement of assistance, which is also the lower bound of a tripartite entanglement measure, the entanglement of collaboration. This may help to characterize the localizable entanglement. Furthermore, an upper bound is also obtained. Deducing from the upper bound of entanglement of assistance, we find a proper form of entanglement monogamy inequality for arbitrary $n$-qubit states, which is analogous to the monogamy constraints for concurrence proposed by Coffman et al. [19] and proven by Osborne and Verstraete [20] for the general case.

The paper is organized as follows. In Sec. II, we derive a lower bound and upper bound of entanglement of assistance for pure tripartite states. In Sec. III, monogamy constraints are proved in terms of this upper bound. Finally in Sec. IV, we conclude with a discussion of our results.

II. BOUND OF ENTANGLEMENT OF ASSISTANCE

We consider a pure $(d_1 \times d_2 \times N)$ tripartite state shared by three parties referred to as Alice, Bob, and Charlie, who performs a measurement on his party to yield a known bipartite entangled state shared by Alice and Bob. Charlie’s aim is to maximize the entanglement of the state between Alice and Bob. This maximum average entanglement that he can create is called entanglement of assistance, which was originally defined in terms of entropy of entanglement $[12,13]$. In this paper, we define entanglement of assistance in terms of the entanglement measure $I$ concurrence $E_A(\psi_{ABC}) = E_A(\rho_{AB}) = \max p C(\langle \phi_i | \psi_{ABC} \rangle)$, which is maximized over all possible pure-state decompositions of $\rho_{AB} = \text{Tr}_C(\psi_{ABC}) = \sum p | \phi_i \rangle \langle \phi_i |$. By applying the method in Ref. [4], we can obtain the lower bound of entanglement of assistance for pure tripartite states.

For any given pure-state decomposition of $\rho_{AB}$: $\rho_{AB} = \sum p | \phi_i \rangle \langle \phi_i |$, we have

$$E_A(\psi_{ABC}) = \max \sum p C(\langle \phi_i | \psi_{ABC} \rangle)$$

$$= \max \sum p_1 \sqrt{\sum \langle \phi_i | S_{mn} | \phi_i \rangle^2}$$

$$\geq \max \sqrt{\sum \langle \phi_i | S_{mn} | \phi_i \rangle^2}$$

where $S_{mn} = L_m \otimes L_n$, $L_m, m = 1, \ldots, d_1 (d_1 - 1)/2$, and $L_n, n = 1, \ldots, d_2 (d_2 - 1)/2$ are the generators of groups SO($d_1$) and SO($d_2$), respectively. The inequality holds according to the Minkowski inequality $\sum_{i=1}^{p} |x_i|^{p/q} \leq |\sum_{i=1}^{p} |x_i|^q|^{p/q}$, $p > 1$.

For any given decomposition $\rho_{AB} = \sum p_i | \phi_i \rangle \langle \phi_i |$, we can take the matrix notation [21] and rewrite the state as $\rho_{AB} = P \Phi P^\dagger$, where $P$ is a diagonal matrix with $P_{ii} = p_i$ and $\Phi$ is a unitary matrix whose columns are the vectors $\phi_i$. Consider the eigenvalue decomposition $\rho_{AB} = \Psi M \Psi^\dagger$, where $M$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\rho_{AB}$ and the columns of the matrix $\Psi$ correspond to the eigenvectors of $\rho_{AB}$. Moreover, we can get the relation $\Psi^{1/2} = \Psi M^{1/2} U$, where $U$ is a unitary matrix. Thus we can rewrite inequality (1) as

$$E_A(\rho_{AB}) \geq \max \sum_{mn} \sum_{i} |P_1^{1/2} S_{mn} P_{mn}^{1/2} | \langle \phi_i |$$

$$= \max \sum_{mn} \sum_{i} | U^T M^{1/2} T S_{mn} \Psi M^{1/2} U_{li} |$$

$$= \max \sum_{mn} \sum_{i} | U^T A_{mn} U_{li} |^2$$

where $A_{mn} = M^{1/2} T S_{mn} \Psi M^{1/2}$. 

We investigate the entanglement of assistance which quantifies capabilities of producing pure bipartite entangled states from a pure tripartite state. The lower bound and upper bound of entanglement of assistance are obtained. In the light of the upper bound, monogamy constraints are proved for arbitrary $n$-qubit states.
In terms of the Cauchy-Schwarz inequality \((\Sigma x_i^2)^{1/2}(\Sigma y_i^2)^{1/2} \geq \Sigma x_i y_i\), the inequality

\[
E_a(\rho_{AB}) \geq \max \sqrt{\left( \sum_{mn} \left( \sum_i U_i T A_{mn} U_i^\dagger \right) i \right)^2 \sum_{mn} z_{mn}^2} = \max \sum_i \left| U_i T (\sum_{mn} z_{mn} A_{mn}) U_i^\dagger \right| \tag{3}
\]

is implied for any \(z_{mn} = y_{mn} \exp(i \theta_{mn})\), with \(y_{mn} \geq 0\) and \(\Sigma y_{mn}^2 = 1\). Since \(\sum_{mn} z_{mn} A_{mn}\) is a symmetric matrix, we can always find a unitary matrix \(U\) such that \(\Sigma_i U_i T (\sum_{mn} z_{mn} A_{mn}) U_i^\dagger = \|[\sum_{mn} z_{mn} A_{mn}]\|\) as shown in Ref. [22], where \(\|\|\) stands for the trace norm defined by \(\|G\| = \text{Tr}(G^2)^{1/2}\).

For an arbitrary unitary matrix \(V\), we have

\[
\sum_i \left| V_i^T (\sum_{mn} z_{mn} A_{mn}) V_i \right|^2 = \sum_i \left| V_i^T (U_i T (\sum_{mn} z_{mn} A_{mn}) U_i^\dagger) V_i \right|^2 = \sum_i \left| (U_i^T V_i)^T \text{Diag}(\lambda_1, \lambda_2, \cdots) (U_i^T V_i)^\dagger \right|^2 \lambda_i = \sum_i \lambda_i,
\]

where \(\lambda_i(z)\)'s, dependent on the choice of the \(y\) and \(\theta\), are the singular values of the matrix \(T = \sum_{mn} z_{mn} A_{mn}\), i.e., the square roots of the eigenvalues of the positive Hermitian matrix \(T T^\dagger\). Therefore, the maximum of Eq. (3) is given by \(\max_z \in C \|[\sum_{mn} \lambda_i(z)]\| = \max_{\lambda_i} \|[\sum_{mn} z_{mn} A_{mn}]\|\). Hence, we arrive at the lower bound of entanglement of assistance for a pure tripartite state as following:

\[
E_a(\rho_{AB}) \geq \max_{\phi_j} \sum_{mn} z_{mn} A_{mn} \tag{4}
\]

Moreover, it has been shown by Gour and Spekkens [23] that, for tripartite states, entanglement of collaboration [24,25] is greater than or equal to entanglement of assistance in terms of a given entanglement measure. Therefore, our lower bound is also the one for entanglement of collaboration. Our bound may help to characterize localizable entanglement. For a pure \(2 \times 2 \times N\) state, this lower bound is consistent with the result of Ref. [16].

Generally, for each set \(\{y_{mn}, \theta_{mn}\}\), we can obtain a lower bound of \(E_a(\rho_{AB})\), which can be tightened by numerical optimization. In fact, there is always one matrix \(A_{mn}\) that gives the main contribution to the right-hand side of Eq. (4). Hence, the singular values of this matrix provide a simple algebraic approximation for the lower bound of \(E_a(\rho_{AB})\). As an example, we consider a state \(|\phi\rangle_{ABC} = |\langle 000| + |\langle 111|\rangle\rangle\rangle\) with \(a+b+c=1\). Without loss of generality, we assume that \(a \geq b \geq c \geq 0\). Then, we get a purely algebraic estimate for the lower bound of \(E_a(\rho_{AB})\), \(2\sqrt{3}/9\), obtained through an optimization.

We can also obtain the upper bound of entanglement of assistance. From the definition of entanglement of assistance, we have

\[
[E_a(\rho_{AB})]^2 = \left[ \max_i \sum \rho_i C(|\phi_i\rangle_{AB}) \right]^2 = \max \sum_i |\langle \phi_i | C(\psi_{i3}) \rangle|^2 \sum_i (\rho_i)^2 = \max_i \sum_i |\langle \phi_i | C(\psi_{i3}) \rangle|^2 \sum_i (\rho_i)^2 \leq 2(1 - \text{Tr}(\rho_{A}^2)),
\]

where \(\rho_i^2 = \text{Tr}(|\psi_{i3}\rangle \langle \psi_{i3}|)\). The first inequality holds according to the Cauchy-Schwarz inequality [26]; the last one, which has also been proved in Ref. [27], holds due to the convex property of \(\text{Tr}(\rho^2_i)\). Define the upper bound as the tangle of assistance \(T_a(\rho_{AB}) = \max \sum \rho_i C(\psi_{i3})\).

In addition to the tangle of assistance being the upper bound of entanglement of assistance, the tangle of assistance for \(n\)-qubit states also exhibits some proper forms of monogamy inequality. In fact, not only does the concurrence satisfy the monogamy constraints for \(n\)-qubit state [19,20], but also the entanglement of assistance exhibit monogamy constraints for \(n\)-qubit pure states [28–30]. We will show below the monogamy inequalities in the light of the tangle of assistance [31].

### III. Monogamy Inequality

Consider a pure tripartite state \(|\Psi\rangle_{ABC}\). The tangle of assistance is defined by \(T_a(|\Psi\rangle_{ABC}) = \max \sum \rho_i \sum_{\psi_i} \rho_i \sum \rho_i \sum_{\psi_i} \sum_{\psi_i} \langle \psi_{i3} | C(\psi_{i3}) \rangle^2 \sum_i \rho_i C(\psi_{i3}) \sum_{\psi_i} \langle \psi_{i3} | \psi_{i3} \rangle \langle \psi_{i3} | \psi_{i3} \rangle \). In the case of pure state \(\rho_{ABC}\), the tangle of assistance is the square of concurrence of this state.

**Theorem 1.** For an arbitrary \(n\)-qubit state, the tangle of assistance satisfies

\[
T_a(\rho_{A_1 A_2} + \rho_{A_1 A_3} + \cdots + \rho_{A_1 A_n}) \geq T_a(\rho_{A_1 (A_2 A_3 \cdots A_n)}),
\]

for arbitrary tripartite states \(\rho_{ABC}\) in \(2 \times 2 \times 2^n\) system.

We first prove Eq. (6) for pure states. In this case, due to the local-unitary invariance of \(T_a(\rho_{ABC})\), we can rotate the basis of subsystem \(C\) into the local Schmidt basis \(|\psi_i\rangle\), \(k = 1, \ldots, 4\), given by the eigenvectors of \(\rho_{C} = \text{Tr}_{AB}(\rho_{ABC})\). In this way, we can regard the \(2^n \times 2^n\)-dimensional qudit as an effective four-dimensional qudit. Therefore, we simply need to prove Eq. (6) for a \(2 \times 2 \times 4\) pure state \(ABC\). For pure states of a tripartite system \(ABC\) of two qubits \(A\) and \(B\) and a four-level system \(C\), we have

\[
T_a(\rho_{ABC}) = T_a(\rho_{AC}) = S_2(\rho_A) - \max \sum \rho_i C(\psi_{i3}) \sum_{\psi_i} \langle \psi_{i3} | \langle \psi_{i3} | \rho_{C} \langle \psi_{i3} | \psi_{i3} \rangle \rangle \sum \rho_i C(\psi_{i3}) = T_a(\rho_{AC}^{\text{eff}}).
\]

It can be shown that any pure-state de-
composition of $\rho_{AB}$ can be realized by positive-operator-valued measures (POVMs) $\{M_j\}$ performed by Bob, the rank of which is 1 (for more details see [17,32]). Therefore, we get the following expression:

$$\tau_\delta(\rho_{AC}) = \max_{\{M_j\}} \sum_{\{p_j\}} p_j S_2(\rho_j),$$

(7)

where the maximum runs over all rank-1 POVMs on Bob’s system, $p_j = \text{Tr}(M_j \otimes M_{\rho_{AB}})$ is the probability outcome $x_j$, and $\rho_j = \text{Tr}_{\{M_j\}}(I \otimes M_{\rho_{AB}}) / p_j$ is the posterior state in Alice’s subsystem. For convenience, we take the definition $I(\rho_{AB}) := S_2(\rho_{A}) - \max \sum p_j S_2(\rho_j)$. By comparing $I(\rho_{AB})$ to Eq. (6) for pure tripartite states, we see that it is sufficient to prove the inequality $I(\rho_{AB}) \leq \tau_\delta(\rho_{AB})$ for all two-qubit states $\rho_{AB}$.

We first derive a computable formula for $I(\rho_{AB})$. Any bipartite quantum state $\rho_{AB}$ may be written as

$$\rho_{AB} = \Lambda \otimes I_B([V_B')(V_B')^\dagger),$$

(8)

where $[V_B',B]$ is the symmetric two-qubit purification of the reduced density operator $\rho_B$ on an auxiliary qubit system $B'$ and $\Lambda$ is a qubit channel from $B'$ to $A$. Deducing from Eq. (7), we have $p_j = \text{Tr}(\Lambda \otimes M_{\rho_{AB}}) / p_j = \text{Tr}(\Lambda \otimes M_{\rho_{AB}}) \times (\Lambda \otimes I_B([V_B')(V_B')^\dagger]) / p_j = \text{Tr}(\Lambda \otimes M_{\rho_{AB}}) / p_j$. Since the rank of $\Lambda$ is 1, $\text{Tr}(\Lambda \otimes M_{\rho_{AB}}) = 0$, $\Lambda$ is a pure state. Moreover, all pure-state decompositions of $\rho_B$ $= \text{Tr}(\Lambda \otimes M_{\rho_{AB}}) = \rho_B$ can be realized by the rank-1 POVM measurements $\{M_j\}$ operated on subsystem $B$ of $[V_B',B]$. Hence, $I(\rho_{AB})$ satisfies

$$I(\rho_{AB}) = S_2(\Lambda(\rho_B)) - \max_{\{p_j,\{\phi_j\}\}} \sum_{\{p_j,\{\phi_j\}\}} p_j S_2(\Lambda(\{\phi_j\})),

(9)

where the maximum runs over all pure-state decompositions $\{p_j,\{\phi_j\}\}$ of $\rho_B$ such that $\Lambda(\rho_B) = \sum p_j \Lambda(\{\phi_j\}) = \rho_B$.

The action of a qubit channel $\Lambda$ on a single-qubit state $\rho = (I + \mathbf{r} \cdot \sigma) / 2$, where $\sigma$ is the vector of Pauli operators, may be written as $\Lambda(\rho) = (I + (\mathbf{L} \cdot \mathbf{r} + 1) / 2)$, where $\mathbf{L}$ is a 3 × 3 real matrix and $\mathbf{L}$ is a three-dimensional vector. In this Pauli basis, the possible pure-state decompositions of $\rho_B$ are represented by all possible sets of probabilities $\{p_j\}$ and unit vectors $\{\mathbf{r}_j\}$ such that $\sum p_j \mathbf{r}_j = \mathbf{r}$, where $(I + \mathbf{r}_j \cdot \sigma) / 2 = \rho_j$. In terms of the Block representation of one-qubit states, the linear entropy $S_2$ is given by $S_2(\mathbf{L}_j \cdot \mathbf{r}_j + 1) = 1 - (\mathbf{L}_j \cdot \mathbf{r} + 1)^2$. In this way, we get the following equation:

$$I(\rho_{AB}) = S_2(\Lambda(\rho_B)) - \max_{\{p_j,\{\phi_j\}\}} \sum_{\{p_j,\{\phi_j\}\}} p_j S_2(\Lambda((I + (\mathbf{r}_j \cdot \sigma) / 2) / 2)) = 1,$n

(10)

without loss of generality, we assume that $\mathbf{L}'^T \mathbf{L}$ is diagonal with diagonal elements $\lambda_i = \lambda_i \leq \lambda_i$. The constraints $|\mathbf{r}_B + \mathbf{x}_j| = 1$ lead to the identities $|\mathbf{x}_j|^2 + 1 - 2|\mathbf{r}_B|^2|\mathbf{x}_j|^2 = |\mathbf{x}_j|^2 - (\mathbf{x}_j \cdot \mathbf{x}_j)|^2$. Substituting this into Eq. (10), we get $I(\rho_{AB}) = \lambda_\delta(1 - |\mathbf{r}_B|^2) + \min_p \lambda_i \sum_j p_j (\lambda_i - \lambda_i)(x_j)(x_j)^2$. This expression is obviously minimized by choosing $x_j = x_j = 0$ for $j$. Then, from the condition $|\mathbf{r}_B + \mathbf{x}_j| = 1$, $\mathbf{x}_j$ have two solutions. The ensemble of two states corresponding to such two solutions can reach the minimum $\lambda_\delta(1 - |\mathbf{r}_B|^2)$. As $S_2(\rho_{AB}) = (1 - |\mathbf{r}_B|^2)$, we obtain the following computable expression:

$$I(\rho_{AB}) = \lambda_\delta S_2(\rho_B).$$

Note that a local filtering operation of the form $\rho_{AB} = \mathbf{T}(I \otimes \mathbf{B}' \rho_B) \rho_{AB}$ leaves L invariant and transforms $S_2(\rho_{AB}) = \rho_{AB}^T \rho_{AB}$. Therefore, the operator $B$ is invertible, we can get the conclusion that there does not exist a pure-state decomposition $\{p_j,\{\phi_j\}\}$ of $\rho_{AB}$ such that $\tau_\delta(\rho_{AB}) > \lambda_\delta B^2$. For the case that the operator $B$ is not invertible, the two pure-state decomposition also does not exist. Furthermore, there exists exactly an optimal pure-state decomposition $\{p_j,\{\phi_j\}\}$ of $\rho_{AB}$ such that $\tau_\delta(\rho_{AB}) = \text{det}(\mathbf{B})^2 / \text{Tr}(I \otimes \mathbf{B}' \rho_B) \rho_{AB}$ by the contradiction. For the case that the operator $B$ is not invertible, such pure-state decomposition does not exist. Furthermore, there exists exactly an optimal filtering operation for which $\rho_B = I$, we can assume, without loss of generality, that $S_2(\rho_B) = 1$.

Let us consider $\rho_{AB}$ with $\rho_B = I$. Then, in terms of Pauli operators, we can rewrite the pure state as follows: $(I \otimes \mathbf{B})|\mathbf{B}'\rangle\langle\mathbf{B}'| / \text{Tr}(I \otimes \mathbf{B}' \rho_B) \rho_{AB} = 1 / I + \sum \rho_{ij} \sigma_i \sigma_j$. Since $\rho_{AB} = (\text{det}(\mathbf{B})^2 / \text{Tr}(I \otimes \mathbf{B}' \rho_B) \rho_{AB})$, it transforms exactly in the same way as the tangle of assistance $\tau_\delta(\rho_{AB})$ does. As there always exists a filtering operation for which $\rho_B = I$, we can assume, without loss of generality, that $S_2(\rho_B) = 1$.

Therefore, the pure-state decomposition $\{p_j,\{\phi_j\}\}$ of $\rho_{AB}$ such that $\tau_\delta(\rho_{AB}) = \lambda_\delta B^2$ transforms exactly in the same way as the tangle of assistance $\tau_\delta(\rho_{AB})$ does. Thus, we always find local-unitary operators, in terms of the theorem of singular value decomposition, so that $U_1 \otimes U_2 \rho_{AB} = \mathbf{R} \rho_{AB} \mathbf{R}^T$, where $P$ and $Q$ are real orthogonal matrices and $\mathbf{L}'$ is a diagonal matrix with its diagonal elements the singular values of $\mathbf{L}$. Because of the local-unitary invariance of $\tau_\delta(\rho_{AB})$ and $I(\rho_{AB})$, without loss of generality, we assume that $\rho_{AB} = (1 + \Sigma \rho_{ij} \sigma_i \sigma_j + \Sigma \rho_{ij} \sigma_i \sigma_j)$, where $R$ is a diagonal matrix with its diagonal elements the singular values of $\mathbf{L}$.

$$\rho_{AB} = \frac{1}{4} \begin{pmatrix} 1 + R + t_3 & 0 & t_1 - i t_2 & R_1 - R_2 \\ 0 & 1 + R + t_3 & R_1 + R_2 & t_1 + i t_2 \\ t_1 + i t_2 & R_1 + R_2 & 1 - R_1 - t_3 & 0 \\ R_1 - R_2 & t_1 + i t_2 & 0 & 1 + R_1 - t_3 \end{pmatrix},$$

the inequality $1 - t_1^2 - t_2^2 - t_3^2 \geq R_3$ must hold. Therefore we obtain
These inequalities imply that $H(\rho_{AB}) \leq \tau_a(\rho_{AB})$ for all two-qubit states $\rho_{AB}$, which then proves Eq. (6) for pure states.

Now we extend Eq. (6) to mixed state case. Consider the maximizing pure-state decomposition $\{p_\alpha, |\phi_\alpha\rangle\}$ for $\tau_a(\rho_{ABC})$. By applying the inequality Eq. (6) and taking into account the concavity of $\tau_a$ we have $\tau_a(\rho_{ABC}) = \sum_\alpha p_\alpha \tau_a(\rho^A_{\alpha}) \leq \sum_\alpha p_\alpha [\tau_a(\rho^A_{\alpha} + \rho^B_{\alpha})] \leq \tau_a(\rho^A + \rho^B), where $\rho^A_{\alpha} = |\phi_\alpha\rangle \langle \phi_\alpha|$. Let $C = C_1 C_2$ be a $2 \times 2^{n-3}$ system and apply Eq. (6), then we get $\tau_a(\rho_{PABC}) \leq \tau_a(\rho_{AB}) + \tau_a(\rho_{AC}) \leq \tau_a(\rho_{AB}) + \tau_a(\rho_{AC}) + \tau_a(\rho_{BC})$. Successively applying Eq. (6) to partitions of $C$, we obtain the inequality Eq. (5) by induction.

In fact, Eq. (5) turns out to be an equality for product states under partition $A |BC; C_n$. For the generalized $n$-qubit GHZ states $|\psi\rangle = (|0\cdots0\rangle + |1\cdots1\rangle) / \sqrt{2}$, Eq. (5) is a strictly inequality.

[25] Entanglement of collaboration quantifies the maximum amount of entanglement that can be generated between two parties, $A$ and $B$, from a tripartite state with collaboration composed of local operations and classical communication among the three parties.

[31] An analogous monogamy inequality for the concurrence of assistance is obtained in Ref. [29]. In terms of the results in Ref. [29], it is easy to prove the theorem 1 for pure n-qubit states. In this paper, we provide a different detailed method to prove the theorem for any n-qubit states.