Interacting domain walls in an easy-plane ferromagnet

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The Landau-Lifshitz equation for an anisotropic (easy-plane) ferromagnet is formulated as a Riemann-Hilbert problem on a Riemann surface of the spectral parameter. Exact multiple domain wall solutions can be obtained in a systematic and exhaustive manner by considering all possible pole arrangements on the Riemann surface. Explicit calculations for up to four poles have been carried out, yielding all possible double wall solutions, including states of colliding walls, breather modes of bound walls, and a set of solutions corresponding to marginally bound walls.

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I. INTRODUCTION

Over the past three decades, an enormous amount of literature has established the importance throughout physics of solitons and the underlying completely integrable models. Although such models are idealizations, they provide otherwise unobtainable analytic insight into nonlinear dynamics, new states about which to perturb, and exact results against which to benchmark approximate and numerical methods. While a variety of techniques for studying integrable models and for discovering soliton solutions exists now, the study of Riemann-Hilbert problems, initially introduced in the 1970's as a reformulation of the inverse scattering problem,[1] has proved especially fruitful in the fields of integrable differential equations and integrable statistical models.[2–5] In this paper, we show that the Riemann-Hilbert approach is the sufficiently more simple and elegant way to construct the multidomain wall states of an anisotropic (easy-plane) ferromagnet. The important point is that these solutions are quite nontrivial, for example, includes moving domain walls.

The dynamics of a ferromagnet can be described by the Landau-Lifshitz equation

$$\frac{\partial \mathbf{M}}{\partial t}=\mathbf{M}\times\left(\frac{\partial^2\mathbf{M}}{\partial x^2}+J\mathbf{M}\right), \tag{1}$$

where $\mathbf{M}(x,t)=(M_x,M_y,M_z)$ is magnetization vector, and $J=\mathrm{diag}(J_x,J_y,J_z)$ describes the magnetic anisotropy. For the most general biaxial ferromagnet with $J_x\neq J_y\neq J_z$, the multisoliton solutions including ones describing bound states and the collision of domain walls has been discussed in Ref. 6. All solutions of uniaxial anisotropic ferromagnet with $J_x=J_y\neq J_z$ can be found as a limit case $J_x\rightarrow J_y$ of solutions for the biaxial ferromagnet in Ref. 6. There is a twofold degeneracy in the ground state for the easy axis case with $J_x=J_y<J_z$. Here we consider the easy-plane case with $J_x=J_y>J_z$, where the ground state has a continuum of degeneracy. This applies to materials such as $CsNiF_3$ and $(C_6H_{11}N\,H_3)Cu\,Br_3$.

In the easy-plane case, the construction of soliton solutions of Eq. (1) has been discussed in many papers.[5,6] Lax representations have been found in Refs. 7,8. The scattering states of colliding walls have been found by the Hirota method.[9] Two-parameter compact soliton solutions have also been constructed by an ansatz[10] and Darboux matrix.[11] The inverse scattering problem of biaxial ferromagnet was considered on the ground of the Riemann-Hilbert problem on a torus.[3,4] For any uniaxial limit case (an easy axial or easy-plane), the torus became degenerate and the standard Riemann surface appears. In this paper, we formulate the Riemann-Hilbert problem on a standard Riemann surface, which can be easily solved to yield multiple parameter families of solitons in a systematic manner.

II. THE RIEMANN-HILBERT METHOD

The Riemann-Hilbert method works for general initial value problems of integrable equations, and reduces into a system of linear algebraic equations for pure soliton solutions which can be solved in closed form.[2–5] Following standard techniques,[7] we introduce the ''lax pair'' whose compatibility equation reproduces Eq. (1):

$$\partial_x\Psi=L\Psi,\qquad \partial_t\Psi=N\Psi, \tag{2}$$

where L and N are 2×2 matrices defined by

$$L=-i\lambda M_z\sigma_x-i\mu M_x\sigma_y-i\mu M_y\sigma_z,$$

$$\begin{aligned}N=&\,i2\mu^2M_z\sigma_x+i2\mu\lambda M_x\sigma_y+i2\mu\lambda M_y\sigma_z-i\lambda(M_x\partial_xM_y\\&-M_y\partial_xM_x)\sigma_x-i\mu(M_y\partial_xM_z-M_z\partial_xM_y)\sigma_y\\&-i\mu(M_z\partial_xM_x-M_x\partial_xM_z)\sigma_z.\end{aligned} \tag{3}$$

The σ's are Pauli matrices. The spectral parameters λ and μ satisfy $\lambda^2=\mu^2+4\rho^2$, where $\rho=\frac{1}{4}(J_x-J_z)^{1/2}$.

Our Riemann-Hilbert method begins by ''rationalizing'' the constraint between the spectral parameters by writing $\lambda=z+\rho^2z^{-1}$ and $\mu=z-\rho^2z^{-1}$, where z can be regarded as the independent (i.e., unconstrained) spectral parameter. Note that the operators L and N are analytic functions of the independent parameter z, except at the singularities $z=0$ and

$z=\infty$. Furthermore, the operators have the following symmetries with respect to discrete transformations in the complex z plane

$$L(z)=-L^{\dagger}(\overline{z})\equiv -T_1L(z),$$

$$L(z)=\sigma_z\overline{L(\rho^2\overline{z}^{-1})}\sigma_z\equiv T_2L(z), \qquad (4)$$

where $L^{\dagger}$ and $\overline{L}$ indicate Hermitian and complex conjugations of L, respectively. The same symmetries are also satisfied by the operator N. These properties imply the solvability of the Riemann-Hilbert problem and associated integral and algebraic equations.[12]

Let us consider the easy-plane ferromagnet with the ground state $M_0=(0,1,0)$. The corresponding Jost solution of the scattering problem may be chosen as $\Psi_0=e^{-i\mu(x-2\lambda t)\sigma_z}$, and the solution of compatibility Eq. (2) is written as $\Psi=\Xi(x,t;z)e^{-i\mu(x-2\lambda t)\sigma_z}$, where Ξ is a 2×2 matrix. Soliton solutions can be generated by studying the Riemann-Hilbert problem for Ξ, which requires the followings.

(i) Ξ and Ξ^{-1} must be analytic in z everywhere including ∞ and 0 except for some poles $\{z_j\}$.

(ii) Ξ must satisfy the symmetries $\Xi T_1\Xi=I$ and $T_2\Xi=\Xi$ to ensure the symmetries for L and N. Such symmetries require that the set of poles $\{z_j\}$ be invariant under reflection about the real axis and inversion about the circle $|z|=\rho$.

(iii) ΞU_j must be analytic at $z=z_j$ for some 2×2 matrix of the form

$$U_j(x,t;z)=e^{-i\mu(x-2\lambda t)\sigma_z}v_je^{i\mu(x-2\lambda t)\sigma_z}, \qquad (5)$$

where v_j is rational in z with a pole at z_j and $\det v_j$ is analytic and nonvanishing at $z=z_j$.

(iv) L obtained through Ξ must have the form $L=zL_1+z^{-1}L_2$, as in the original Lax pair.

Some explanations are in order. First, the matrices v_j and the points z_j define the discrete part of scattering data and thus are naturally associated with solitons. We will consider the case of two poles corresponding to a single domain wall and the case of four poles or two double poles corresponding to double domain walls explicitly. Second, the unitary "twist" in Eq. (5) ensures that any Jost solution Ψ satisfies the Lax pair (2) for some L and N rational in z with poles at 0 and ∞ of the correct order. Third, one can readily show that condition (iii) remains invariant if v_j is replaced by $v_j'\equiv v_jw_j$ for some $w_j=w_j(z)$ analytic with nonvanishing determinant at $z=z_j$. We use this invariance to define an equivalence relation "$\sim$" which we write as $v'\sim v$. Then the symmetry condition (ii) requires that $T_1v_1\sim v_2^{-1}$, $T_1v_2\sim v_1^{-1}$, $T_2v_j\sim v_j$. Finally, the condition (iv) is necessary because a general solution of the first three conditions can yield a L that has a z^0 term which does not exist in the original definition (3).

We now outline the procedure for the solution of our Riemann-Hilbert problem. We first choose a certain number of poles and the scattering matrices v_j according to the symmetry requirements. Then the solution Ξ is determined by the conditions (i), (ii), and (iii) up to a left matrix multiplier $\Gamma(x,t)$ independent of z. Thus we can proceed with the unique solution $\xi(x,t;z)$ which satisfies (i), (ii), (iii), and a normalization condition $\xi(x,t;\rho)=I$. Once such a ξ has been found, we can apply condition (iv) to determine the multiplier $\Gamma(x,t)$ to obtain $\Xi=\Gamma\xi$.

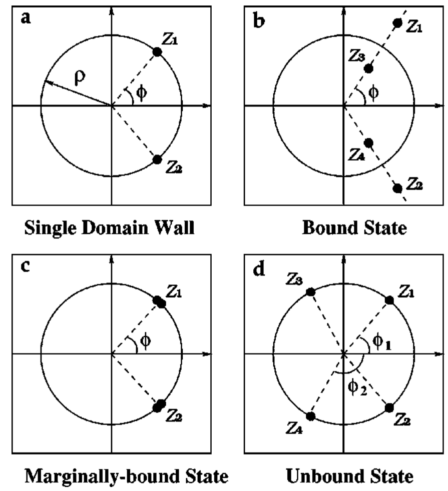

FIG. 1. Poles for different domain wall solutions.

In the following, we show how the multiplier Γ and the final solutions for the magnetization can be expressed in terms of ξ at particular values of z. First, the symmetries for both Ξ and ξ imply that $\Gamma=e^{i\omega(x,t)\sigma_x}$ with $\omega(x,t)$ being a real scalar function. Then the Jost function can be written as $\Psi=e^{i\omega\sigma_x}\xi e^{-i\mu(x-2\lambda t)\sigma_z}$, and the L operator can be obtained from the first of Eqs. (2) as

$$L=e^{i\omega\sigma_x}(\partial_x\xi\xi^{-1}-i\mu\xi\sigma_z\xi^{-1}+i\partial_x\omega\sigma_x)e^{-i\omega\sigma_x}. \qquad (6)$$

Because ξ is analytic everywhere except for some poles, the first two terms inside the bracket in the above expression must be analytic everywhere except for $z=0$ and $z=\infty$, and must have the following analytic form:

$$\partial_x\xi\xi^{-1}-i\mu\xi\sigma_z\xi^{-1}=Q_0+z^{-1}Q_1+zQ_2, \qquad (7)$$

where the coefficients Q_j are matrices independent of z. They can be expressed in terms of ξ at particular values of z by comparing the two sides of the last equality. By taking $z\to\infty$ and $z\to 0$, we find $Q_2=-i\xi(\infty)\sigma_z\xi^{-1}(\infty)$ and $Q_1=i\xi(0)\sigma_z\xi^{-1}(0)$, respectively. Similarly, by taking $z=\rho$ and $z=-\rho$, we obtain $Q_0=\frac{1}{2}\partial_x\xi(-\rho)\xi^{-1}(-\rho)$. Finally, the lack of z^0 term in the L operator as specified in condition (iv) demands that $i\partial_x\omega\,\sigma_x+Q_0=0$. Utilizing the above result for Q_0, this equation can be easily solved to yield $e^{-i2\omega\sigma_x}=\xi(-\rho)$.

III. SINGLE DOMAIN WALL

In the absence of any pole, Ξ must be independent of z and the solution is the ground state. The case of a single pole (or any odd number of poles) is ruled out because it is impossible to satisfy both symmetries T_1 and T_2. The simplest nontrivial case is two simple poles z_1 and z_2 on the circle $|z|=\rho$ with $z_1=\overline{z_2}=\rho e^{i\phi}$ as shown in Fig. 1(a), which gives the well known solution of one domain wall. As a demonstration of how to proceed with the Riemann-Hilbert method, we first find out this simplest nontrivial solution.

We choose the scattering matrices with the triangular form

$$v_1=\begin{pmatrix} 1 & \dfrac{b_1}{z-z_1} \\ 0 & 1 \end{pmatrix}, \quad v_2=\begin{pmatrix} 1 & 0 \\ \dfrac{b_2}{z-z_2} & 1 \end{pmatrix}, \tag{8}$$

where $b_2=-\overline{b_1}$ and $\overline{b_j}/b_j=\overline{z_j}/z_j$ because of the symmetries. This special triangular form[13] ensures that the solution approaches the ground state at $x=\pm\infty$. The twisted scattering matrices U_j have the same form as above, except that the coefficients b_j are replaced by $c_j=b_j e^{i2\mu(z_j)[x-2\lambda(z_j)t]}$.

It follows from (i) and the normalization of ξ at $z=\rho$ that the solution $\xi(x,t;z)$ must have the form

$$\xi(z)=I+(z-\rho)\left(\frac{A}{z-z_1}+\frac{B}{z-z_2}\right), \tag{9}$$

where the coefficients A and B can then be determined by condition (ii). Now the matrices $\xi(\infty)$ and $\xi(-\rho)$ can be computed from Eq. (7), the scalar function ω from Eq. (9), and finally the solution can be written as

$$M_x=-\cos\phi\,\mathrm{sech}\zeta,\quad M_y=\tanh\zeta,\quad M_z=\sin\phi\,\mathrm{sech}\zeta, \tag{10}$$

where $\zeta(x,t)=4\rho\sin\phi(x-Vt)+\alpha$, $V=4\rho\cos\phi$, $\phi=\arg z_1$, and α is an integration constant. It describes a moving domain wall with velocity V, width $(|4\rho\sin\phi|)^{-1}$, and energy $E=8\rho\sin\phi$. It include two limits: a Neél wall with maximum velocity 4ρ for $\phi=0$, and Bloch wall with maximum energy 8ρ for $\phi=\pi/2$.

IV. DOUBLE DOMAIN WALLS

More interesting solutions are double domain walls, which are found with four poles in the complex plane. There are only two possible arrangements for four poles, as shown in Figs. 1(b) and 1(d).

The first case is achieved by allowing a pole off circle $|z|=\rho$, say, $z_1=re^{i\phi}$, with $r>\rho$. Then symmetries T_1 and T_2 demand that the three others be located at $z_2=\overline{z_1}$, $z_3=(\rho^2/r)e^{i\phi}$, $z_4=\overline{z_3}$. Following the procedure of our Riemann-Hilbert method, we find the solution for the case of four poles off circle

$$M_x=\frac{2\sin\phi}{\Delta}[\sin\phi\sinh\zeta_1\cos\zeta_2+f\cos\phi\cosh\zeta_1\sin\zeta_2],$$

$$M_y=\frac{1}{\Delta}[\cos(2\phi)+\sinh^2\zeta_1-g\sin^2\phi\sin^2\zeta_2],$$

$$M_z=\frac{2\sin\phi}{\Delta}[\cos\phi\cosh\zeta_1\cos\zeta_2-f\sin\phi\sinh\zeta_1\sin\zeta_2], \tag{11}$$

where $f=(r^2+\rho^2)/(r^2-\rho^2)$, $g=4\rho^2r^2/(r^2-\rho^2)^2$, and

$$\Delta=\cosh^2\zeta_1+g\sin^2\phi\sin^2\zeta_2,$$

$$\zeta_1=2(r+\rho^2/r)\sin\phi\,(x-V_1t)+\alpha_1,$$

$$\zeta_2=2(r-\rho^2/r)\cos\phi\,(x-V_2t)+\alpha_2, \tag{12}$$

with $V_1=4[(r^4+\rho^4)/r(r^2+\rho^2)]\cos\phi$ and $V_2=2(r+\rho^2/r)[\cos(2\phi)/\cos\phi]$. The two parameters α_1 and α_2 are integration constants.

The above solutions describe the bound states of double domain walls. The two walls oscillate against each other, while their center of mass moves with constant velocity V_1. The ratio r/ρ tells how tightly the double walls are bounded. When the four poles are far off circle, i.e., $r/\rho\gg 1$, the double domain walls are so tightly bounded together that they can be viewed as one traveling solution whose shape is modulated by a spin wave with velocity V_2. This is the reason that this solution is called soliton solution historically.[11] As the four poles get closer to the circle $r/\rho\to 1$ the oscillation of double domain walls begin to become obvious. Its amplitude gets larger while its frequency $|V_1-V_2|$ gets smaller. The shapes of double domain walls are well kept except around collision time, which is $t=0$ if constants α_1 and α_2 are taken to be zero.

The other possible situation for four poles is shown in Fig. 1(d), where the four poles are on the circle $|z|=\rho$. Let two of them to be at $z_1=\rho e^{i\phi_1}$ and $z_3=\rho e^{i\phi_2}$, then the other two are located at $z_2=\overline{z_1}$ and $z_4=\overline{z_3}$, as demanded by symmetries T_1 and T_2. We find the solution in this case

$$M_x=\frac{1}{\Delta}[\cos\phi_2(\cos\phi_2-\cos\phi_1)\sinh\zeta_1+\cos\phi_1(\cos\phi_1-\cos\phi_2)\sinh\zeta_2],$$

$$M_y=\frac{1}{\Delta}[\sin^2\phi_1+\sin^2\phi_2-\sin\phi_1\sin\phi_2\cosh\zeta_1\cosh\zeta_2+(1-\cos\phi_1\cos\phi_2)(\sinh\zeta_1\sinh\zeta_2-1)],$$

$$M_z=\frac{1}{\Delta}[\sin\phi_2(\cos\phi_1-\cos\phi_2)\cosh\zeta_1+\sin\phi_1(\cos\phi_2-\cos\phi_1)\cosh\zeta_2], \tag{13}$$

where

$$\Delta=(1-\cos\phi_1\cos\phi_2)\cosh\zeta_1\cosh\zeta_2-\sin\phi_1\sin\phi_2(1+\sinh\zeta_1\sinh\zeta_2),$$

$$\zeta_1=4\rho\sin\phi_1\,(x-V_1t)+\alpha_1,$$

$$\zeta_2=4\rho\sin\phi_2\,(x-V_2t)+\alpha_2, \tag{14}$$

with $V_1=4\rho\cos\phi_1$ and $V_2=4\rho\cos\phi_2$, while α_1 and α_2 are integration constants.

This solution depicts a pair of scattering domain walls. The scattering process is completely similar to the elastic collision of two balls with same mass. Before collision, they move towards each other, one with velocity V_1, the other with V_2. At the collision, they exchange velocities. Then they move apart, one with V_2, the other with V_1.

As we have exhausted the situations for four poles, we have not exhausted all possible double domain wall solutions of the Landau-Lifshitz equation. There is one more double domain wall solution corresponding to the case shown in Fig. 1(c), where we have two poles of the second order. In this case, the two scattering matrices take the form

$$v_1=\begin{pmatrix} 1 & \dfrac{b_{11}}{z-z_1}+\dfrac{b_{12}}{(z-z_1)^2} \\ 0 & 1 \end{pmatrix},$$

$$v_2=\begin{pmatrix} 1 & 0 \\ \dfrac{b_{21}}{z-z_2}+\dfrac{b_{22}}{(z-z_2)^2} & 1 \end{pmatrix} \tag{15}$$

and

$$\xi(z)=I+(z-\rho)\left(\frac{A}{z-z_1}+\frac{B}{z-z_2}+\frac{C}{(z-z_1)^2}+\frac{D}{(z-z_2)^2}\right). \tag{16}$$

Using our Riemann-Hilbert method with these matrices, we find the solution for the case of two poles of the second order

$$M_x=\frac{2\sin\phi}{\Delta}[\sin\phi\sinh\zeta_1+\zeta_2\cos\phi\cosh\zeta_1],$$

$$M_y=\frac{1}{\Delta}[\cos(2\phi)+\sinh^2\zeta_1-\zeta_2^2\sin^2\phi],$$

$$M_z=\frac{2\sin\phi}{\Delta}[\cos\phi\cosh\zeta_1-\zeta_2\sin\phi\sinh\zeta_1], \tag{17}$$

where

$$\Delta=\cosh^2\zeta_1+\zeta_2^2\sin^2\phi,$$

$$\zeta_1=4\rho\sin\phi\,(x-V_1t)+\alpha_1,$$

$$\zeta_2=4\rho\cos\phi(x-V_2t)+\alpha_2, \tag{18}$$

with $V_1=4\rho\cos\phi$ and $V_2=4\rho[\cos(2\phi)/\cos\phi]$, while α_1 and α_2 are integration constants.

In this solution, the double domain walls also collide head-on and then depart from each other forever. However, unlike the previous case, the relative velocities of the two walls tend to zero at long times as $1/t$. We call this solution a marginally bound state of the domain walls. It can be worked out from Eq. (17) that the relative position between the two walls is $\Delta X=(1/2\rho\sin\phi)\ln[|4(V_1-V_2)\sin(2\phi)|t]$. It means that the effective potential between the two well-separated domain walls decreases exponentially as $-e^{-8x\rho\sin\phi}$. We note that this marginally bound state can be viewed as the limiting case of either a bound state or an unbound state. Taking either $r\to\rho$ in Eq. (11) or $\phi_1\to\phi_2$ in Eq. (13) reproduces the solution (17).

V. CONCLUSION

These three solutions, the bound state, the unbound state, and the marginally bound state, are all possible double domain wall solutions. This claim is intuitively obvious from a physical point of view, and now it is also clear mathematically with the help of the Riemann-Hilbert method. All other pole distributions, besides the four cases shown in Fig. 1, correspond to solutions of more than double domain walls. For example, two poles of third order give us a solution of triple domain walls which collide and depart from each other with diminishing velocities; a pair of poles on circle and four poles off circle produce a solution describing a single wall scattering with a double-wall bound state. In principle, the Riemann-Hilbert method can give us all possible solutions of the Landau-Lifschitz equation, and all possible pole arrangements in the complex plane give us all domain wall solutions.

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