# Quantum Tunneling of Bose-Einstein Condensates in Optical Lattices under Gravity

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We investigate the quantum tunneling of Bose-Einstein condensates in optical lattices under gravity in the "Wannier-Stark localization" regime and "Landau-Zener tunneling" regime. Our results agree with experimental data [B. P. Anderson *et al.,* Science **282**, 1686 (1998); F. S. Cataliotti *et al.,* Science **293**, 843 (2001)]. We obtain the total decay rate which is valid over the entire range of temperatures, and show how it reduces to the appropriate results for the classical thermal activation at high temperatures, the thermally assisted tunneling at intermediate temperatures, and the pure quantum tunneling at low temperatures. We design an experimental protocol to observe this new phenomenon in further experiments.

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The decay of metastable states involving the quantum tunneling is important in many areas of contemporary physics, including the Landau-Zener tunneling in semiconductor superlattices [1], in ultracold atom systems [2], and in Bose-Einstein condensates [3], the Josephson tunneling [4–7], and the quantum tunneling across spin domains [8] in Bose-Einstein condensates. Recently, Zhang *et al.* employed the periodic instanton method to investigate Josephson tunneling of Bose-Einstein condensates confined in double-well potential traps at zero temperature [6]. In this Letter, we examine the different Bose-Einstein condensates — one in optical lattices under gravity in the regime of "Wannier-Stark localization"— in which metastable states can exhibit interesting decay behavior distinct from that seen in the Landau-Zener regime. Using the periodic instanton method [9], we calculate the total decay rate in the Wannier-Stark regime for all temperatures. Except for the explained results in experiments [4,5], we design an experimental protocol to observe this phenomenon.

In recent $^{87}$Rb experiments [4,5], Bose-Einstein condensates were confined in an array of optical traps under gravity. Weakly interacting Bose-Einstein condensates were trapped in 30 wells [4] or 200 wells [5]. Each well contained approximately $10^3$ condensed atoms, with the peak densities matching a Gaussian profile. Since the array is oriented vertically, the atoms undergo coherent motion driven by the interwell gravitational potential. Even at a peak density of $n_0 = 10^{13}$ cm$^{-3}$, the mean field energy $U_{\rm MF}$ is $k_B \cdot 4$ nK per atom at the maximally populated well, while the kinetic energy per atom approximates $k_B \cdot 157$ nK [4]. Neglecting atomic interactions, the chemical potential difference between adjacent traps is determined by the gravitational potential $mgz$, where $z$ is the vertical coordinate. A combined optical plus gravitational trapping potential is illustrated in Fig. 1 of Ref. [4]. The corresponding Hamiltonian is $H = \frac{P^2}{2m} + U_l(x,y)\sin^2(\frac{2\pi}{\lambda}z) + mgz$, where $P$ is the momentum operator, $m$ is the atomic mass, $\lambda$ is the wavelength of light used to confine atoms, and $U_l(x,y)$ is the depth of each well which is determined by the transverse intensity profile of the laser beams. This is a periodic sloping potential in which the tilt is proportional to gravity $g$.

We will distinguish two different tunneling regimes of Bose-Einstein condensates in optical lattices under gravity: (i) tunneling between the spatially delocalized states in different Bloch bands, usually called Landau-Zener tunneling which is only the adiabatic approximation to the regime [4,5]; (ii) tunneling between the spatially localized states in different individual wells, corresponding to tunneling between Wannier-Stark localized states. Of course, in the absence of any external force such as gravity, the tunneling between the localized states in the wells is what creates the Bloch bands, which survive for weak external forces. Thus, for a small external force the system is in the Landau-Zener tunneling regime, whereas for a large external force the system shifts to the Wannier-Stark regime. Experimentally, the tunneling which related Landau-Zener tunneling has been measured [4,5]. However, to date tunneling in the Wannier-Stark regime has not been investigated, either theoretically or experimentally.

*Wannier-Stark tunneling.*— For appropriate experimental parameters, each lattice state can have a potentially significant tunneling probability into the continuum, and it can be modeled as a point emitter of de Broglie waves with an emission rate proportional to the tunneling probability. The output from an array of such emitters is obtained by summing over the coherent emission from each well. For a close to experimental case [4,5], we focus on the parameter region in which atoms decay from a localized state of one well in optical lattices under gravity, and consider tunneling of atoms from a metastable state with finite energy $E$ through an effective potential barrier $V(z)$. To guarantee the existence of the well, we require $mg < \frac{4\pi}{\lambda}U_l(x,y)$.

  

Defining $\Delta$ as in Fig. 1 of Ref. [4], the condition $\Delta > 0$ means that there is no crossing; the atoms can decay to infinity via tunneling, where $\Delta = \Delta_{\rm top} - [U_l(x,y) - E]$, $\Delta_{\rm top} = mg\lambda/2$ is the difference between tops of two adjacent maxima of the optical lattices under gravity, and $E$ is the energy of any metastable state. The excited metastable state will reduce to a metastable ground state when $E = E_0$, where $E_0$ is the energy of the lowest metastable state, as illustrated by heavy lines in Fig. 1 of Ref. [4]. In this case, we can approximate the optical lattices under gravity by an effective potential $V(z)$.

The decay rate $\Gamma$ of the quantum metastable states is defined as the imaginary part of the complex energy $E$, $\Gamma = \frac{2}{\hbar}\,{\rm Im}E$. Let $|\Psi\rangle$ denote an eigenstate of the Hamiltonian $H$ with energy $E$; we consider the transition amplitude $A$ from the state $|\Psi\rangle$ to itself—the "survival probability" of $|\Psi\rangle$—in the presence of quantum tunneling over Euclidean time $2\beta$,

$$A = \langle\Psi|e^{-2H\beta}|\Psi\rangle = e^{-2E\beta}, \tag{1}$$

where $\beta = iT$, with $T$ the temperature. We will first calculate this defined transition amplitude $A$ using the path integral method [10] for periodic instanton. Then, comparing the defined amplitude $A$ in Eq. (1) with the calculated amplitude $A$ of the following Eq. (2), we can find the decay rate $\Gamma$.

In quantum field theories, the periodic instanton method is an efficient tool to study quantum tunneling of metastable states. The periodic instanton represents the pseudocondensed atom configuration responsible for tunneling under the barrier at energy $E$. Tunneling out of a metastable state in a potential $V(z)$ can be treated as motion with imaginary time $\tau = it$ in the corresponding inverted potential, where $V(z) = V_0 + \frac{1}{2}U_l(x,y)\cos\theta_0(\frac{4\pi}{\lambda}z - \pi + \theta_0)^2 + \frac{1}{3}U_l(x,y)\sin\theta_0(\frac{4\pi}{\lambda}z - \pi + \theta_0)^3$, $V_0 = -U_l(x,y)\cos\theta_0 + U_l(x,y)(\pi - \theta_0)\sin\theta_0$, and $\theta_0 = \arcsin(\frac{mg\lambda}{4\pi U_l(x,y)})$. The Euler-Lagrange equations in Euclidean space-time lead to $\frac{1}{2}m(\frac{dz}{d\tau})^2 - V[z(\tau)] = -E$. The corresponding "periodic instanton" solution $z_p(\tau) = z_3 + (z_2 - z_3)\,{\rm sn}^2(u|k)$ is periodic with period $\mathcal{T}$, and $z_p(\tau + \mathcal{T}) = z_p(\tau)$, where $z_1(E) > z_2(E) > z_3(E)$ denote three roots of the equation $V(z) = E$, and $z_2$ and $z_3$ denote the turning points of the instanton motion in inverted potential. This case corresponds to that of a special coordinate with periodic boundary conditions [9], while ${\rm sn}(u|k)$ denotes a Jacobian elliptic function with the modulus $k = \sqrt{\frac{z_2 - z_3}{z_1 - z_3}}$, where $u = \gamma(k')\tau$, $\gamma(k') = \sqrt{\frac{32\pi^3 U_l(x,y)\cos\theta_0(z_1 - z_3)}{3m\lambda^3(\pi - 2\theta_0)}}$, and $k' = \sqrt{1 - k^2}$ is the complementary modulus of $k$.

For the previous periodic instanton solution, the Feynman propagator of the path integral is divergent because the velocity of instanton vanishes at the turning points $z_2$ and $z_3$. This singularity must be smoothed out by turning point integrations of $dz_2$ and $dz_3$ with $\tau_2 - \tau_3 = 2\beta$. Further, we replace the wave functions $\psi_E(z_2)$ and $\psi_E(z_3)$ by their leading WKB approximations, and expand the action in powers of $z_2 - z_p(\beta)$ up to the second order, which corresponds to the one loop approximation. Using the one loop expansion of the action and completing the turning point integration, we can obtain one instanton contribution to the transition amplitude $A_1$. Similarly, we can obtain $j$th instanton contribution $A_j$, etc.

In the dilute gas approximation, all instantons were totally independent; the total transition amplitude $A$ is given by the summation over all the instanton contributions

$$A = e^{-2E\beta - i\frac{\beta}{\mathcal{K}_1(k')}\sqrt{\frac{8\pi^2 U_l(x,y)\cos\theta_0(z_1 - z_3)}{3m\lambda^2}}\,e^{-W}}, \tag{2}$$

where

$$W = \frac{64k^4}{15\hbar}\sqrt{\frac{2\pi m}{3\lambda}\,U_l(x,y)\cos\theta_0}\,(z_1 - z_3)^{5/2} \times [(1 - k^2)(k^2 - 2)\mathcal{K}_1(k) + 2(k^4 - k^2 + 1)\mathcal{K}_2(k)], \tag{3}$$

where $\mathcal{K}_1(k)$ and $\mathcal{K}_2(k)$ are the complete elliptic integrals of the first and second kinds.

Generally, the imaginary part of energy $E$ is obtained by comparing calculated amplitude $A$ in Eq. (2) with defined amplitude A in Eq. (1). We find that the decay rate of metastable state with energy $E$ can be written as

$$\Gamma = \frac{\omega_E}{\hbar\mathcal{K}_1(k')}\,e^{-W}, \tag{4}$$

where $\omega_E = \sqrt{\frac{2(z_1 - z_3)}{3}}\,\omega_0$ is the energy dependent frequency. It describes the reduction of the attempt frequency, i.e., the frequency of small oscillations $\omega_0 = \sqrt{\frac{16\pi^2 U_l(x,y)\cos\theta_0}{m\lambda^2}}$ around the metastable minimum. We emphasize that this compact formula is valid for the entire region of energy $0 < E < V_{\rm max}$, where $V_{\rm max} = V_0 + \frac{1}{6}U_l(x,y)\cos\theta_0\cot^2\theta_0$. It can be applied for any excited states from the bottom to the top of the well.

For energies far below the barrier maximum $E \ll V_{\rm max}$, introducing the harmonic approximation $E_n = n\hbar\omega_0 + E_0$, we can evaluate the decay rate $\Gamma_n$ of $n$th low excited state,

$$\Gamma_n = \frac{1}{n!}\left(\frac{432V_{\rm max}}{\hbar\omega_0}\right)^n\Gamma_0\,. \tag{5}$$

When $n = 0$, $\Gamma_n$ reduces to the decay rate $\Gamma_0$ of the metastable ground state with energy $E_0$,

$$\Gamma_0 = 12\omega_0\sqrt{\frac{3V_{\rm max}}{2\pi\hbar\omega_0}}\,e^{-\frac{36V_{\rm max}}{5\,\hbar\omega_0}}. \tag{6}$$

We find that the decay rate depends on the depth of the well. By adjusting the depth of the optical wells $U_l(x,y)$, we could control the tunneling rate from the well so that it was fast enough to observe atoms leaving the traps, but slow enough to allow for direct observation of many periods of the temporally modulated signal before the traps were depleted. The tunneling rate could be increased by lowering the well depths, producing fewer pulses with more atoms per pulse. We hope to observe this interesting phenomenon in further experiments.

We find the tunneling rate of the Landau-Zener regime $\Gamma_{\rm LZ} = \frac{mg\lambda}{4\pi\hbar} e^{-g_c/g}$, where $g_c = \frac{\lambda\epsilon^2}{8\hbar^2}$ is a critical acceleration, and $\epsilon$ is the energy gap between the first and second bands. In recent $^{87}$Rb experiments [4,5], $\epsilon \sim E_R$ is the energy gap between the ground state band and continuum states, where $E_R = \frac{2\pi^2\hbar^2}{m\lambda^2}$ is the recoil energy. We find that the Landau-Zener tunneling rate will not depend on the depth of the well, and it is constant in recent experiments [4,5].

It is very interesting to compare the decay rate $\Gamma_0$ of the Wannier-Stark regime with $\Gamma_{\rm LZ}$ of the Landau-Zener regime for recent experiments [4,5]. For $^{87}$Rb atoms with Yale experimental parameters $\lambda = 850$ nm, $U_l(x,y) = 2.1E_R$, we find that $\Gamma_0 = 12.26 \times 10^3$ s$^{-1}$, $\Gamma_{\rm LZ} = 12.37$ s$^{-1}$. The lifetime corresponding to the Landau-Zener regime is 88 ms, and it is very close to lifetime 50 ms observed in the Yale experiment [4]. For $^{87}$Rb atoms with INFM (Istituto Nazionale di Fisica della Materia, Italy) experimental parameters $\lambda = 795$ nm, $U_l(x,y) = 5E_R$, we find that $\Gamma_0 = 2.63 \times 10^3$ s$^{-1}$, $\Gamma_{\rm LZ} = 2.60$ s$^{-1}$. The lifetime corresponding to the Landau-Zener regime is 0.39 s, and it is very close to the lifetime 0.3 s observed in the INFM experiment [5]. In both cases, atoms in the Wannier-Stark regime decay faster 1000 times than those in the Landau-Zener regime. Up to now, quantum tunneling in the Wannier-Stark regime has not been investigated by experiment.

*Temperature dependence.*—Taking a statistical average of the decay rates of different metastable states, we obtain the temperature dependence of the decay rate,

$$\Gamma(T) = \Gamma_0(1 - e^{-\frac{\hbar\omega_0}{k_B T}}) e^{\frac{432 V_{\max}}{\hbar\omega_0} e^{-\hbar\omega_0/k_B T}}, \qquad (7)$$

where $k_B$ is the Boltzmann constant. For $^{87}$Rb atoms with Yale experimental parameter $U_l(x,y) = 2.1E_R$, Fig. 1 shows three distinct regions with different behavior for $\Gamma(T)$. At high temperature, the decay is purely thermally activated and exhibits the expected Arrhenius law [11], $\Gamma_{\rm AR} = \frac{\omega_0}{2\pi} e^{-V_{\max}/k_B T}$. At intermediate temperature, we observe "thermally assisted" tunneling, in which the atom is thermally excited to a range of excited quantum states still trapped in the well and the decay then proceeds from these excited states by quantum tunneling. At a still lower temperature, we enter the regime of pure quantum tunneling from the trapped ground state, which is controlled by the "vacuum" instanton trajectory. Hence we can define the "crossover" temperature, $T_{\rm cr} = \frac{\hbar\omega_0}{2\pi k_B}$, below which quantum tunneling dominates. The decay rate is an independent temperature when below $T_{\rm cr}$, and it will increase with temperature raising when above

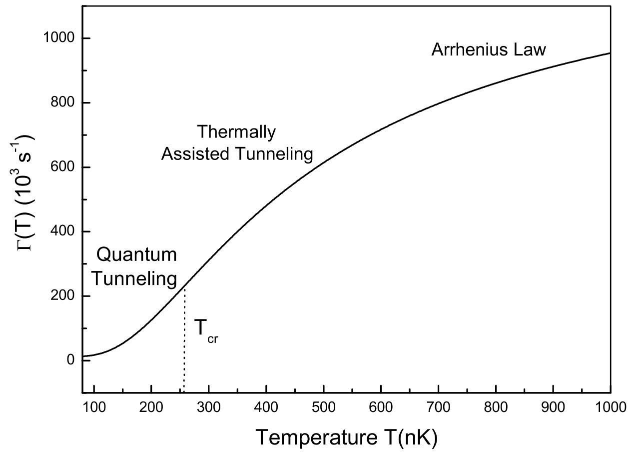


FIG. 1. The temperature dependence of the decay rate $\Gamma(T)$ for $^{87}$Rb atoms with Yale experimental parameters $\lambda = 850$ nm, $U_l(x,y) = 2.1E_R$, where $E_R = \frac{2\pi^2\hbar^2}{m\lambda^2}$ is the recoil energy.

$T_{\rm cr}$. We find that $T_{\rm cr} = 257$ nK for the Yale experiment parameter $U_l(x,y) = 2.1E_R$.

In order to make clearer the meaning of $\Gamma(T)$, we have an initial thermal distribution of the atoms; the population should decay as

$$N(t) = \frac{N_0}{Z_0} \sum_n e^{-\Gamma_n t - \frac{E_n}{k_B T}}, \qquad (8)$$

where $N_0$ is the initial population of trapped atoms. Since $N(t)$ is a sum of exponential decay factors, it does not decay exponentially itself; i.e., it is not a straight line if we plot $\log N(t)$ as a function of time. There is not a single slope, and therefore it cannot be assigned a single decay rate. At long times, the curve approaches a straight line, with the corresponding rate given by $\Gamma_0$; i.e., the slowest channel dominates the long time behavior. In fact, the initial slope at time $t = 0$ corresponds to a rate of $\Gamma(T)$ as calculated, meaning that $\Gamma(T)$ is simply the initial rate of decay of the population. The above discussion is valid if there is no thermalization of the population during decay, which is a situation that the experimentalist can realize. On the other hand, it is also possible to intentionally introduce intrawell transitions between the different levels, such that thermalization can be established in a time $t_c$. In this case, if $t_c$ is short compared to $1/\Gamma(T)$, the population will decay at $\Gamma(T)$ at all times. If it is larger than $1/\Gamma(T)$, the population will decay initially nonexponentially until time $t_c$, after which the population will decay at $\Gamma(T')$, where $T'$ is some temperature below $T$. There is a cooldown of the population due to the initial decay.

To test these predictions in experiments, one must prepare the atoms with the proper protocols, which we will now describe. To measure the tunneling from the lowest metastable state, one must do the following: (i) turn on a potential which has only one state in each well; (ii) accelerate the potential in such a way that only the band of states from these levels are swept along with the potential, leaving all the higher states behind (so they can be neglected); (iii) increase the amplitude of the potential, so that different wells become isolated from each other; (iv) tilt the potential (by acceleration) to achieve the Wannier-Stark regime described by the present theory; and (v) observe how many atoms survive in time $t$.

The time scales in processes (iii) and (iv) should be long enough so that nonadiabatic excitation in each well is negligible, but at the same time short enough [in the case of (iv)] that tunneling has not started yet. In the Wannier-Stark regime, this is easy to realize, as the decay rate is exponentially small in the lattice constant, $\lambda_l$ (the barrier thickness is a substantial fraction of $\lambda_l$), while the level spacing in the well is only inversely proportional to $\lambda_l$ (the well size is also comparable to the lattice constant).

To measure the decays from excited states and at higher temperatures, we need to prepare a thermal distribution of atoms in the wells by using the following experimental protocol: (i′) starting with a thermal distribution of free atom states, turn on the potential to some amplitude, so that eventually there are $n$ bands lying in the wells; (ii′) accelerate the potential so that the $n$ bands are taken along with the wells, leaving atoms in higher bands behind. The acceleration must be such that the occupation number of each of the $n$ bands is not changed during this process; (iii′)–(v′) the same as (iii)–(v) above. The $n$ bands become $n$ metastable states. The time scales in these steps must be sufficiently adiabatic that the occupation of each band (level) is not changed.

The distribution of occupancies of the metastable states is determined by that of the $n$ bands at the end of step (i′). The latter is in turn determined by how the potential is turned on. We consider three cases, in which the occupation of the $n$ bands can be calculated. (a) If the potential is turned on adiabatically, the occupation of the $n$th band is the same as the occupation of the $n$th Brillouin zone of the free states. (b) If the potential is turned on suddenly, the occupation of the $n$th band is given by the integral of the overlap of a free state $K$ with the band over the thermal distribution of the free state. In both cases, we assume that dissipation is negligible. (c) In the case where the potential is turned on and one waits until the dissipative processes thermalize the atoms in the potential, the occupation of the $n$th band is Boltzmannian with respect to the band energies.

However, in each case, the resulting distribution of occupancies will generally not be Boltzmannian with respect to the energies of the metastable states. Nevertheless, we may modify (ii′) to achieve a Boltzmann distribution of these states. After thermalizing with the potential as described in case (c), we accelerate so that only the lowest bands in the wells are kept. These bands lie near the bottom of the wells and are very narrow in energy. Moreover, the potential near the bottom can be approximated by a harmonic potential; these bands have energies in the form $n\hbar\omega_0 + E_0$. Their occupational distribution is therefore of the form $e^{-n(\hbar\omega_0/k_BT)}$.

After steps (iii′)–(v′), this distribution becomes that of the metastable states. If the bottom of the well of the metastable states is quadratic with a harmonic frequency $\omega$, then this distribution is also thermal with respect to the energies of the metastable states, with an effective temperature given by $T = \frac{\omega}{\omega_0} T_0$.

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