Evolution equation of entanglement for bipartite systems

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We explore how entanglement of a bipartite system evolves when one subsystem undergoes the action of an arbitrary noisy channel. It is found that the dynamics of entanglement of such system is determined by the channel's action on the maximally entangled state, which includes as a special case the results for two-qubit systems [Konrad *et al.*, Nat. Phys. **4**, 99 (2008)]. In particular, for multiqubit or qubit-qudit systems, we get a general factorization law for the evolution equation of entanglement, with one qubit being subject to a noisy channel.

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INTRODUCTION

In quantum-information theory, entanglement is a vital resource for some practical applications such as quantum cryptography, quantum teleportation, and quantum computation [1,2]. To fulfill such tasks by constructing suitable quantum devices, we inevitably encounter some interactions of the multiparticle quantum state under consideration with its environment. These undesired couplings give rise to decoherence, which degrades the entanglement when the particles propagate or the computation evolves. Therefore, it is of great practical importance to investigate the dynamics of entanglement for the quantum systems under the influence of decoherence.

Recently much effort has been devoted to understanding the dynamics of entanglement [3–8]. Instead of deducing the evolution of entanglement from the time evolution of the state, Konrad *et al.* [3] provided a direct relationship between the initial and final entanglement of an arbitrary bipartite state of two qubits with one qubit subject to incoherent dynamics, where the qubit represents the state of the two-dimensional quantum system. It is also discussed in [4] for two-qudit systems, with either system undergoing an arbitrary physical process, where the qudit denotes the state of a D-dimensional quantum system. On the condition that the pure initial state has d nonzero Schmidt coefficients, evolution equality is satisfied during an initially finite time interval. At later times, one has to rely on a hierarchy of entanglement monotones $C_k[|\psi\rangle]$, in which G concurrence is the last member of the hierarchy $G_d[|\psi\rangle]=C_d[|\psi\rangle]$ [9].

In fact, for practical applications in quantum-information processing, multipartite entanglement is often concerned, e.g., cluster states used as a resource for one-way quantum computing [10], multiphoton entangled states [11], etc. For multiqubit systems, the global entanglement [12] and residual entanglement [13] are composed of bipartite entanglement measures. Moreover, bipartite systems with higher dimension can improve the performance of various quantum-information and computation tasks, such as quantum cryptography [14]. In this Brief Report, we investigate the evolution of bipartite entanglement for multipartite pure states with one part of the system undergoing the action of an arbitrary noisy channel, which represents the influence of an environment, of measurements, or of both. In the following, we find that the dynamics of entanglement for bipartite systems under the influence of a noisy channel is determined by the channel's action on the maximally entangled state, instead of exploring the time-dependent action of the channel on all initial states. Therefore, just as Konrad *et al.* [3] first proposed, the robustness of two-qubit entanglement-based quantum-information processing protocols is thus easily and fully characterized by a single quantity, the quantity of entanglement for the evolved maximally entangled state can also to some extent help us to characterize the robustness of bipartite systems. As applications we discuss two examples in detail: the entanglement evolution for a generalized three-qubit W state [15] with one qubit undergoing the action of a generalized amplitude damping channel, and that for the ground state in a nuclear magnetic resonance (NMR) system with one subsystem subject to a decay channel.

EVOLUTION EQUATION OF ENTANGLEMENT

We first define the entanglement measure for bipartite systems. For a pure state $|\chi\rangle=\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}A_{ij}|ij\rangle\in\mathbb{C}^{N_1}\otimes\mathbb{C}^{N_2}$ in the computational basis $|i\rangle$ and $|j\rangle$ of Hilbert spaces $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively, we define the concurrence matrix $\mathbf{C}$ with entries $C_{\alpha\beta}=\langle\chi|(L_\alpha\otimes L_\beta)|\chi^*\rangle$, where $|\chi^*\rangle$ is the complex conjugate of $|\chi\rangle$, L_α, $\alpha=1,\ldots,N_1(N_1-1)/2$, and L_β, $\beta=1,\ldots,N_2(N_2-1)/2$, are the generators of the SO (N_1) and SO (N_2) groups respectively. The Frobenius norm of $\mathbf{C}$ is just the I concurrence [16], $C[|\chi\rangle]=\|\mathbf{C}[|\chi\rangle]\|_F=\sqrt{\sum_{\alpha=1}^{N_1(N_1-1)/2}\sum_{\beta=1}^{N_2(N_2-1)/2}|C_{\alpha\beta}|^2}$, which reduces to the concurrence when restricted to $2\otimes 2$ systems [17]. These quantities can be measured for pure states [18]. As we know, the I concurrence is equal to the length of the concurrence vector [19,20].

The $N_1(N_1-1)/2$ generators L_α of SO (N_1) have the form $L_{\alpha_{(kl)}}=(-1)^{k+l+1}|k\rangle\langle l|+(-1)^{k+l}|l\rangle\langle k|$, $k<l$, where $\alpha_{(kl)}=(j_1,j_2,\ldots,j_{N_1-2})$ with $1\leqslant j_1<j_2<\cdots<j_{N_1-2}\leqslant N_1$, $k<l$,

and $k,l \notin \{j_1,j_2,\ldots,j_{N_1-2}\}$. Therefore, one has $C_{\alpha_{kl}\beta_{k'l'}} = 2(-1)^{k+l+k'+l'}(A_{kk'}A_{ll'}-A_{kl'}A_{lk'})^*$ and $C[|\chi\rangle] = 2\sqrt{\sum_{i<j}^{N_1}\sum_{k<l}^{N_2}|A_{ik}A_{jl}-A_{il}A_{jk}|^2}$, which is just the generalized concurrence in [21] up to a constant factor. $C[|\chi\rangle]$ is zero when $|\chi\rangle$ is separable, i.e., $A_{ij}=b_ic_j$ for some complex numbers b_i, c_j. On the other hand, C takes its maximum value $\sqrt{2(N-1)/N}$ with $N=\min(N_1,N_2)$, when $|\chi\rangle$ is a maximally entangled state.

For a bipartite mixed state $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$, $p_i\geqslant 0$, $\sum_i p_i=1$, the concurrence is defined by the convex roof

$$C[\rho] = \min \sum_i p_i C[|\psi_i\rangle], \tag{1}$$

where $C[|\psi_i\rangle]$ is the norm of the concurrence matrix $\mathbf{C}[|\psi_i\rangle]$ and the minimum is obtained over all possible pure state decompositions $|\psi_i\rangle$ of ρ.

Let $|\chi\rangle$ be the bipartite pure initial state, and let the second subsystem undergo the action of a noisy channel. We will denote the noisy channel by $\mathcal{S}$ thereafter. Then the final state of the system takes the form $\rho'=(\mathbf{1}\otimes\mathcal{S})|\chi\rangle\langle\chi|$. To investigate the properties of the entanglement of the final state ρ', it is convenient to reexpress the initial state as: $|\chi\rangle=(M_\chi\otimes\mathbf{1})|\phi\rangle$, where $M_\chi=\sqrt{N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}A_{ij}|i\rangle\langle j|$ is the filtering operation [22] acting on the first subsystem of a maximally entangled pure state, $|\phi\rangle=\sum_{n=1}^{N_2}|n\rangle\otimes|n\rangle/\sqrt{N_2}$, and $\mathbf{1}$ is the $N_2\times N_2$ identity matrix. Due to the fact that M_χ and $\mathcal{S}$ act on the first and second subsystems of the state $|\phi\rangle$, respectively, the evolution of $|\chi\rangle$ takes the form $\rho'=(M_\chi\otimes\mathbf{1})\rho_{\mathcal{S}}(M_\chi^\dagger\otimes\mathbf{1})$, where $\rho_{\mathcal{S}}=(\mathbf{1}\otimes\mathcal{S})|\phi\rangle\langle\phi|$.

When the channel $\mathcal{S}$ is unitary or stochastic quantum operation given by a local filter, $\rho_{\mathcal{S}}$ is a pure state. For the stochastic quantum operation (i.e., probability nonpreserving), we replace, for convenience, $(\mathbf{1}\otimes\mathcal{S})\rho$ by the normalized state $(\mathbf{1}\otimes\mathcal{S}')\rho$, where $\mathcal{S}'$ has taken into account the normalization of probability for channel $\mathcal{S}$ on state ρ. Then we have the following theorem in terms of concurrence matrix: when $\rho_{\mathcal{S}'}$ is a pure state, the concurrence for the state ρ' is given by

$$C[\rho'] = \sqrt{\sum_{\alpha=1}^{N_1(N_1-1)/2}\sum_{\beta=1}^{N_2(N_2-1)/2}|C'_{\alpha\beta}|^2}, \tag{2}$$

where $C'_{\alpha\beta}=(N_2/2)\sum_{\gamma=1}^{N_2(N_2-1)/2}C_{\alpha\gamma}[|\chi\rangle]C_{\gamma\beta}[\rho_{\mathcal{S}'}]$. $C_{\alpha\gamma}[|\chi\rangle]$ and $C_{\gamma\beta}[\rho_{\mathcal{S}'}]$ are the entries of the concurrence matrices $\mathbf{C}(|\chi\rangle)$ and $\mathbf{C}(\rho_{\mathcal{S}'})$, respectively.

Let us now prove this theorem. Suppose the pure state $\rho_{\mathcal{S}}$ has the generic form, $\rho_{\mathcal{S}'}=|\psi\rangle\langle\psi|$, $|\psi\rangle=\sum_{i,j=1}^{N_2}a_{ij}|ij\rangle$. The final state has the form $\rho'=|\psi'\rangle\langle\psi'|$, where $|\psi'\rangle=\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}B_{ij}|ij\rangle$ and $B_{ij}=\sqrt{N_2}\sum_{l=1}^{N_2}A_{il}a_{lj}$. Some straightforward algebra yields the equation $C'_{\alpha_{kl}\beta_{k'l'}}=2(-1)^{k+l+k'+l'}(B_{kk'}B_{ll'}-B_{kl'}B_{lk'})^* = (N_2/2)\sum_{\gamma=1}^{N_2(N_2-1)/2}(C_{\alpha_{kl}\gamma}[|\chi\rangle]C_{\gamma\beta_{k'l'}}[\rho_{\mathcal{S}'}])$. Hence the concurrence takes the form (2). ■

Remark. With respect to the relations $C[\rho'] = C[|\chi\rangle]C[\rho_{\mathcal{S}}]$ in [3] for two-qubit systems ($N_1=N_2=2$), here we have a similar relation for the corresponding concurrence matrices, $\mathbf{C}[\rho']=(N_2/2)\mathbf{C}[|\chi\rangle]\mathbf{C}[\rho_{\mathcal{S}'}]$.

For a general channel $\mathcal{S}$, the state $\rho_{\mathcal{S}}$ is usually a mixed one. Assume $\rho_{\mathcal{S}}$ has an optimal pure state decomposition $\rho_{\mathcal{S}}=\sum_i p_i|\phi_i\rangle\langle\phi_i|$ such that $C[\rho_{\mathcal{S}}]=\sum_i p_iC[|\phi_i\rangle]$. By convexity we have $C[(\mathbf{1}\otimes\mathcal{S})|\chi\rangle\langle\chi|]=C[\sum_i p_i(M_\chi\otimes\mathbf{1})|\phi_i\rangle\langle\phi_i|(M_\chi^\dagger\otimes\mathbf{1})] \leqslant \sum_i p_iC[(M_\chi\otimes\mathbf{1})|\phi_i\rangle\langle\phi_i|(M_\chi^\dagger\otimes\mathbf{1})]$. According to the Cauchy inequality, we have $|\sum_{\gamma=1}^{N_2(N_2-1)/2}C_{\alpha\gamma}[|\chi\rangle]C_{\gamma\beta}[\rho_{\mathcal{S}}]|^2 \leqslant \sum_{\gamma=1}^{N_2(N_2-1)/2}|C_{\alpha\gamma}[|\chi\rangle]|^2\sum_{\gamma'=1}^{N_2(N_2-1)/2}|C_{\gamma'\beta}[\rho_{\mathcal{S}}]|^2$. In terms of Eq. (2) we get

$$\begin{aligned} C(\rho') &\leqslant \sum_i p_iC[(M_\chi\otimes\mathbf{1})|\phi_i\rangle\langle\phi_i|(M_\chi^\dagger\otimes\mathbf{1})] \\ &\leqslant \frac{N_2}{2}C[|\chi\rangle]\sum_i p_iC[|\phi_i\rangle] \\ &= \frac{N_2}{2}C[|\chi\rangle]C[\rho_{\mathcal{S}}]. \end{aligned} \tag{3}$$

This inequality can be generalized to the case that the initial state ρ_0 is a mixed one,

$$C[(\mathbf{1}\otimes\mathcal{S})\rho_0] \leqslant \frac{N_2}{2}C(\rho_0)C[\rho_{\mathcal{S}}]. \tag{4}$$

If we consider bipartite states in $N_1\otimes 2$ system, the result (2) can be generalized for arbitrary noisy channels $\mathcal{S}$. So we get the following corollary: if the pure initial state is a bipartite $N_1\otimes 2$ one, while the second subsystem is subject to an arbitrary noisy channel $\mathcal{S}$, we have the following evolution equation of concurrence:

$$C[\rho'] = C[|\chi\rangle]C[\rho_{\mathcal{S}}]. \tag{5}$$

This corollary is proved as follows. Without loss of generality, we suppose that $\rho_{\mathcal{S}}$ is a mixed state. By using the procedure of the optimal pure state decomposition adopted in Ref. [17], there must exist an optimal pure state decomposition for a $2\otimes 2$ mixed state

$$\rho_{\mathcal{S}} = \sum_i p_i|\psi_i\rangle\langle\psi_i|, \tag{6}$$

such that $C[\rho_{\mathcal{S}}]=C[|\psi_i\rangle]$, $\forall\ i$, is satisfied. Suppose the pure decomposition Eq. (6) is not optimal for concurrence of ρ' in terms of Eq. (1). Then there must exist another decomposition other than Eq. (6), $\rho_{\mathcal{S}}=\sum_i q_i|\psi_i'\rangle\langle\psi_i'|$, which is an optimal pure state decomposition of $\rho'=(M_\chi\otimes\mathbf{1})\rho_{\mathcal{S}}(M_\chi^\dagger\otimes\mathbf{1})$. In terms of Eq. (2), we have

$$\begin{aligned} C[\rho'] &= \sum_i q_iC[(M_\chi\otimes\mathbf{1})|\psi_i'\rangle] \\ &= C[|\chi\rangle]\sum_i q_iC[|\psi_i'\rangle] \geqslant C[|\chi\rangle]C[\rho_{\mathcal{S}}]. \end{aligned} \tag{7}$$

However, in terms of the optimal pure state decomposition (6) and convexity, we have

$$\begin{aligned} C[\rho'] &< \sum_i p_iC[(M_\chi\otimes\mathbf{1})|\psi_i\rangle] \\ &= C[|\chi\rangle]\sum_i p_iC[|\psi_i\rangle] = C[|\chi\rangle]C[\rho_{\mathcal{S}}]. \end{aligned} \tag{8}$$

It contradicts with Eq. (7). Therefore, the optimal pure state

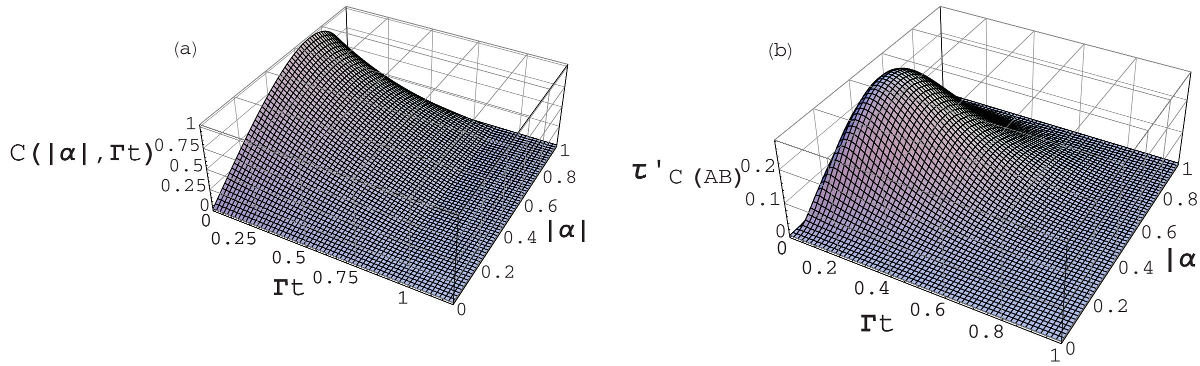

FIG. 1. (Color online) (a) Concurrence $C_{AB:C}(|\alpha|,t)$ vs Γt and amplitude $|\alpha|$, where Γ is the generalized amplitude decay rate. The systems of AB and of C disentangle completely and abruptly in just a finite time for all $|\alpha|$ in the range shown. (b) Dependence of the residual entanglement $\tau'_{C(AB)}$ on Γt and amplitude $|\alpha|$ for $|\beta|=|\gamma|$. The residual entanglement gets its maximized value at the point $\Gamma t=0.0936$ and $|\alpha|=0.4996$.

decomposition (6) is also optimal for concurrence (1) of ρ'. Therefore, we get Eq. (5). ∎

The result (5) can also be generalized to the case that the initial state ρ_0 is mixed. Let $\rho_0=\Sigma_i p_i|\psi_i\rangle\langle\psi_i|$ be an optimal pure state decomposition, in the sense that the average Frobenius norm of the concurrence matrix over this pure state decomposition is minimal. According to convexity, we have $C[(\mathbf{1}\otimes\mathcal{S})\rho_0]=C[\Sigma_i p_i(\mathbf{1}\otimes\mathcal{S})|\psi_i\rangle\langle\psi_i|]\leqslant\Sigma_i p_i C[(\mathbf{1}\otimes\mathcal{S})|\psi_i\rangle\langle\psi_i|]$. Using Eq. (5), we have

$$C[(\mathbf{1}\otimes\mathcal{S})\rho_0]\leqslant C[\rho_0]C[\rho_{\mathcal{S}}]. \tag{9}$$

When $N_1=2$, the results (5) and (9) reduce to the main result of Konrad *et al.* [3]. The results (2),(3)–(5) and (9) show that the dynamics of entanglement for bipartite systems under a one-sided noisy channel is determined by the channel's action on the maximally entangled state.

Let us study in what case the evolution equation of entanglement holds for $2\otimes 2$ pure initial state under the influence of local two-sided channel $\mathcal{S}_1\otimes\mathcal{S}_2$. If the Kraus operators [23] of $\mathcal{S}_1$, $K_{\mathcal{S}_1}$, satisfies the condition $\mathrm{Tr}(\sigma_x K_{\mathcal{S}_1})=\mathrm{Tr}(\sigma_y K_{\mathcal{S}_1})=0$, we have $C[(\mathcal{S}_1\otimes\mathcal{S}_2)\chi]=C[(\mathcal{S}_1\otimes\mathcal{S}_2)|\phi\rangle\langle\phi|]C(|\chi\rangle)$ for any initial state $|\chi\rangle=a|00\rangle+b|11\rangle$ in the Schmidt expression. For instance, the above condition can be satisfied for the phase noise channel $\mathcal{S}_p$ with Kraus matrix $K_1=\left(\begin{smallmatrix}\nu & 0\\ 0 & 1\end{smallmatrix}\right)$, $K_2=\left(\begin{smallmatrix}\omega & 0\\ 0 & 0\end{smallmatrix}\right)$, where the time-dependent parameters $\nu=\exp[-\Gamma t]$ and $\omega=\sqrt{1-\nu^2}$. If $\mathcal{S}_1$ satisfies the condition $\mathrm{Tr}(\sigma_z K_{\mathcal{S}_1})=\mathrm{Tr}(K_{\mathcal{S}_1})=0$, for $|\chi\rangle=a|00\rangle+b|11\rangle$, we have $C[(\mathcal{S}_1\otimes\mathcal{S}_2)\chi]=C[(\mathcal{S}_1\otimes\mathcal{S}_2)|\phi\rangle\langle\phi|]C(|\chi\rangle)$; for $|\chi\rangle=a|01\rangle+b|10\rangle$, we have $C[(\mathcal{S}_1\otimes\mathcal{S}_2)\chi]=C[(1\otimes\mathcal{S}_2\mathcal{S}_1)|\phi\rangle\langle\phi|]C(|\chi\rangle)$.

APPLICATION TO TWO REALISTIC SYSTEMS

Let us consider a three-qubit system with the third qubit exposed to the *generalized amplitude damping* channel, $\mathcal{S}_{\mathrm{GAD}}$, describing the effect of dissipation to an environment at finite temperature, which is the case relevant to NMR quantum computation. The channel usually adopts the form $K_0=\sqrt{p}\left(\begin{smallmatrix}1 & 0\\ 0 & \nu\end{smallmatrix}\right)$, $K_1=\sqrt{p}\left(\begin{smallmatrix}0 & \omega\\ 0 & 0\end{smallmatrix}\right)$, $K_2=\sqrt{1-p}\left(\begin{smallmatrix}\nu & 0\\ 0 & 1\end{smallmatrix}\right)$, $K_3=\sqrt{1-p}\left(\begin{smallmatrix}0 & 0\\ \omega & 0\end{smallmatrix}\right)$. Without loss of generality, set $p=\frac{1}{2}$. We study how the residual entanglement evolves [24]. For the initial state $|\chi\rangle_{ABC}=\alpha|001\rangle+\beta|010\rangle+\gamma|100\rangle$, the generalized W state, we obtain $C_{AB:C}(|\alpha|,t)=C[(\mathbf{1}\otimes\mathbf{1}\otimes\mathcal{S}_{\mathrm{GAD}})\rho_{AB:C}]=|\alpha|\sqrt{1-|\alpha|^2}(\exp[-2\Gamma t]+2\exp[-\Gamma t]-1)$ in terms of Eq. (5). As shown in Fig. 1(a), over a continuous range of $|\alpha|$ values, $C_{AB:C}(|\alpha|,t)$ actually goes abruptly to zero in a finite time and remains zero thereafter. This is the "entanglement sudden death" effect [7,25,26]. The residual entanglement is calculated as $\tau'_{C(AB)}=C^2_{AB:C}-C^2_{BC}-C^2_{AC}$. For simplicity, we focus on $|\beta|=|\gamma|$. For a certain value of $|\alpha|$, it demonstrates that the total entanglement of the composite system A-C and of the composite system B-C decreases more rapidly than that of AB-C during the initial time. In the context of quantum cryptography, if we considered the third party C as an eavesdropper [27], it indicates that the amount of information that an eavesdropper could potentially obtain about the secret key to be extracted goes abruptly to zero and remains zero thereafter just as in the entanglement sudden death effect. As an example, it also shows that the multipartite entanglement measure defined in $\tau_{C(AB)}$ [28] is not a monotone [29] for mixed three-qubit states. In fact the residual entanglement provides bounds on the distribution of private correlations and quantifies the frustration of entanglement between different parties as proposed in [27]. The residual entanglement is considered as a measure to quantify three-way entanglement only through the convex-roof definition [30].

The evolution of entanglement is basically related to the Hamiltonian of a physical system. For example, consider the ground state of an NMR system, $|\psi\rangle=\sqrt{1/3}|02\rangle-\sqrt{1/3}|11\rangle+\sqrt{1/3}|20\rangle$, in which the coupling Hamiltonian of two spin-1 nuclei can be expressed as: $\mathcal{H}=J\hat{S}_1\hat{S}_2$, $J>0$, where $\hat{S}_1$ and $\hat{S}_2$ are spin operators of nuclei 1 and 2, respectively. Supposing that $|\psi\rangle$ is the initial state, under the single-sided relaxation operation $M=\mathrm{diag}(e^{-\Gamma_2 t},e^{-\Gamma_1 t},1)$, $\Gamma_2\geqslant\Gamma_1>0$ [31], the final state becomes $\rho'=(\mathbf{1}\otimes M)|\psi\rangle\langle\psi|(\mathbf{1}\otimes M^{\dagger})/p$, where $p=\mathrm{Tr}[(\mathbf{1}\otimes M)|\psi\rangle\langle\psi|(\mathbf{1}\otimes M^{\dagger})]$. In terms of Eq. (2), we obtain the dependence of entanglement on t, $c[\rho']=\sqrt{4[(e^{-2\Gamma_1 t}+1)(e^{-2\Gamma_2 t}+1)-1]}/(1+e^{-2\Gamma_1 t}+e^{-2\Gamma_2 t})$. When $t\gg 1/\Gamma_1$, $\rho'\to|20\rangle$.

CONCLUSIONS

In summary, we have investigated the time evolution of entanglement for arbitrary bipartite systems, with one part subject to interactions with environments. Explicit expressions are derived for bipartite systems and a general factorization law is obtained for multiqubit or qubit-qudit systems with one qubit undergoing the action of a noisy channel. It allows one to know the time evolution of entanglement for an arbitrary initial state, if one knows the time evolution of entanglement for the bipartite maximally entangled state. The latter depends only on the detailed noisy channel and has nothing to do with the initial states. Our results can be used to infer the evolution of entanglement under certain time-continuous influences of the environment. Due to the fact that all the quantities of entanglement measure can be evaluated efficiently for pure states, it can help the experimental characterization of entanglement dynamics. Moreover, the results can be also directly applied to input-output processes such as gates used in sequential quantum computing. As applications we have studied the entanglement evolution of the generalized three-qubit W state, with one qubit undergoing the action of generalized amplitude damping channel. We also obtain the evolution of entanglement for the ground state in an NMR system with one subsystem subject to a decay channel.

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