

Necessary and sufficient conditions for local creation of quantum correlation

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Quantum correlation can be created by a local operation from some initially classical states. We prove that the necessary and sufficient condition for a local trace-preserving channel to create quantum correlation is that it is not a commutativity-preserving channel. This condition is valid for arbitrary finite dimension systems. We also derive the explicit form of commutativity-preserving channels. For a qubit, a commutativity-preserving channel is either a completely decohering channel or a mixing channel. For a three-dimensional system (qutrit), a commutativity-preserving channel is either a completely decohering channel or an isotropic channel.

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Quantum correlation is the unique phenomenon of quantum physics and believed to be a resource for quantum-information processes, which can generally surpass the corresponding classical schemes. Many previous studies focus on entanglement, a well-known quantum correlation, because of its apparent role in teleportation, superdense coding [1,2], etc. Recently measures of the nonclassicalness of correlation, such as quantum discord [3] and quantum deficit [4,5], have begun to attract much attention since the discovery that some quantum-information schemes can be realized without entanglement but with a positive quantum discord [6,7]. Much progress has been made to quantify the amount of quantum correlation in different physical systems [8,9] and to give it intuitive and operational interpretations. It is shown that quantum discord can be operationally interpreted as the minimum information missing from the environment [10]. One-way quantum deficit [11,12] has been found as the reason for entanglement irreversibility [13] and can be related to quantum entanglement via an interesting scheme [14,15].

Quantum noise usually plays a destructive role in quantum information process. However, there are situations that local quantum noise can enhance nonlocal quantum properties for some mixed quantum states. For example, local amplitude damping can increase the average teleportation fidelity for a class of entangled states [16–18]. Quantum discord can also be increased or created by local noise [19–21]. An interesting result is that any separable state with positive quantum discord can be produced by local positive operator-valued measure on a classical state in a larger Hilbert space [22]. In fact, almost all states in the Hilbert space contain quantum correlation, and an arbitrary small disturbance can drive a classical state into a quantum state with nonzero quantum correlation [23]. Counterintuitively, it has recently been discovered that mixedness is as important as entanglement for quantum correlation. In particular, some mixed states contain more quantum discord than that of a maximally entangled pure state when the dimension of the system is large enough [15]. Thus it is of interest to know how the effect of mixedness on the quantum correlation of quantum states. The condition for local increase of quantum correlation has been derived for the qubit case [24], and it has been pointed out that this condition is not valid for high-dimension systems.

In this article we derive a simple necessary and sufficient condition for a local channel to create quantum correlation in some half-classical states, which is valid for arbitrary finite dimension systems. A trace-preserving local channel can create quantum correlation if and only if it is not a commutativity-preserving channel. For the qubit case we show that a commutativity-preserving channel is either a mixing channel or a completely decohering channel. This confirms the result in Ref. [24]. For the qutrit case, quantum correlation can be created by a local channel in some half-classical input states if and only if the channel is neither a completely decohering channel nor an isotropic channel. We also analyze the reason for a local mixing channel to create quantum correlation in a qutrit situation and then give a conjecture to extend the result of qutrits to arbitrary finite-dimension systems.

The total correlation between two quantum systems is composed of classical and quantum correlations. From this point of view, quantum correlation is defined as the difference between total and classical correlations. Therefore, various measures of quantum correlation defined for one party of a composite system vanish for exactly the same class of states, called half-classical states. Because classical correlation is defined by the correlation that can be revealed by local measurements, a state ρ_{AB} is half classical on B if and only if there exists a measurement on B that does not affect the total state. As proved in Ref. [25] a half-classical state on B can be written as

$$\rho_{AB} = \sum_i p_i \rho_A^{\alpha_i} \otimes |\alpha_i\rangle_B \langle \alpha_i|, \quad (1)$$

where $\{|\alpha_i\rangle_B\}$ consist of an orthogonal basis for the Hilbert space of subsystem B , and $\rho_A^{\alpha_i}$ are corresponding density matrices of A . The subsystem A can be a single quantum particle or an ensemble of quantum particles. In the following, by quantum correlation, we mean quantum correlation defined on subsystem B . The main purpose of this paper is to characterize the channel Λ_B satisfying

$$I_A \otimes \Lambda_B(\rho_{AB}) \in \mathcal{D}_0 \quad \forall \rho_{AB} \in \mathcal{D}_0, \quad (2)$$

where \mathcal{D}_0 is the set of half-classical states. Before providing the condition, we first introduce a class of quantum channels, which we call commutativity-preserving channels.

Definition 1 (commutativity-preserving channel). A commutativity-preserving channel Λ^{CP} is the channel that can

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preserve the commutativity of any input density operators; i.e.,

$$[\Lambda^{\text{CP}}(\xi), \Lambda^{\text{CP}}(\xi')] = 0 \quad (3)$$

holds for any density operators satisfying $[\xi, \xi'] = 0$.

It is worth mentioning an equivalent definition of a commutativity-preserving channel. A channel Λ is a commutativity-preserving channel if and only if

$$[\Lambda(\phi), \Lambda(\psi)] = 0 \quad (4)$$

holds for any pure states satisfying $\langle \phi | \psi \rangle = 0$. The ‘‘only if’’ part is obtained directly by choosing $\xi = |\phi\rangle\langle\phi|$ and $\xi' = |\psi\rangle\langle\psi|$. Conversely, if Eq. (4) holds, by writing ξ and ξ' on their common eigenbasis, we arrive at Eq. (3).

Now we are ready to prove the first main result of this paper. It holds for arbitrary finite-dimension systems.

Theorem 1. A channel Λ acting on subsystem B can create quantum correlation between subsystems A and B for some input half-classical state ρ_{AB} if and only if it is not a commutativity-preserving channel.

Proof. Any separable state can be written as

$$\xi_{AB} = \sum_i p_i \xi_i^A \otimes \xi_i^B, \quad (5)$$

where ξ_i^A are linearly independent. We will first prove that ξ_{AB} is a half-classical state if and only if

$$[\xi_i^B, \xi_j^B] = 0 \quad \forall i, j. \quad (6)$$

For proving the ‘‘only if’’ part, we notice that for any half-classical state, there exists a measurement basis $\Pi_B^{\alpha_j}$ that does not affect the state. Therefore,

$$\sum_i p_i \xi_i^A \otimes (\xi_i^B - \Pi_B^{\alpha_j} \xi_i^B \Pi_B^{\alpha_j}) = 0. \quad (7)$$

Because ξ_i^A are linearly independent, ξ_i^B is diagonal on $\{\Pi_B^{\alpha_j}\}$ and thus satisfies Eq. (6). Conversely, if Eq. (6) holds, ξ_i^B and ξ_j^B have eigenvectors in common for any i and j . By choosing these eigenvectors as the basis for von Neumann measurement, the state does not change after the measurement, which means that ξ_{AB} is a half-classical state. Now consider an arbitrary half-classical state in the form of Eqs. (5) and (6) as the input state, the channel Λ acting on subsystem B leads the state to $\xi'_{AB} \equiv I_A \otimes \Lambda_B(\xi_{AB}) = \sum_i p_i \xi_i^A \otimes \Lambda(\xi_i^B)$, which is still a half-classical state if and only if

$$[\Lambda(\xi_i^B), \Lambda(\xi_j^B)] = 0, \quad (8)$$

for arbitrary choice of ξ_i^B and ξ_j^B satisfying Eq. (6). This is just the definition of a commutativity-preserving channel. Therefore, the channel Λ can create quantum correlation for some input half-classical states if and only if it is not a commutativity-preserving channel. This completes the proof.

The rest of this paper is devoted to exposing the exact form of a commutativity-preserving channel.

Since $[I, \rho] = 0, \forall \rho$, we obtain a necessary condition for a commutativity-preserving channel

$$[\Lambda(I), \Lambda(\rho)] = 0 \quad \forall \rho. \quad (9)$$

When B is a qubit, Eq. (9) is also the sufficient condition. The reason is as follows. By using the linearity of Λ , the left-hand

side of Eq. (4) can be written as

$$\begin{aligned} & \frac{1}{2} [\Lambda(|\phi\rangle\langle\phi| + |\psi\rangle\langle\psi|), \Lambda(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)] \\ & = \frac{1}{2} [\Lambda(I), \Lambda(u\sigma^z u^\dagger)], \end{aligned} \quad (10)$$

where $|\psi\rangle = u|0\rangle$ and $|\phi\rangle = u|1\rangle$. Since any qubit state ρ can be decomposed as $\rho = (I + n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)/2$, Eq. (4) is equivalent to Eq. (9). From this observation, we can see that a qubit channel Λ is commutativity-preserving if and only if it is one of the following two cases:

Case 1. $\Lambda(I) = I$, which means that Λ is a unital channel. Here we define a mixing channel Λ^M as

$$S(\Lambda^M(\rho_S)) \geq S(\rho_S) \quad \forall \rho_S, \quad (11)$$

where $S(\rho) \equiv -\text{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy. It is worth mentioning that when a channel is a mixing channel, its extension to larger systems $I_A \otimes \Lambda_S^M$ is still a mixing channel. As proved in Ref. [26], a mixing channel is equivalent to a unital channel.

Case 2. $\Lambda(I) \neq I$. Then the diagonal basis of $\Lambda(I)$ is specified. According to Eq. (8), the two matrices $\Lambda(\rho)$ and $\Lambda(I)$ share eigenvectors. In other words, the channel Λ takes any input state ρ to a diagonal form on the eigenbasis of $\Lambda(I)$ and is thus a completely decohering channel.

Therefore, when B is a qubit, a commutativity-preserving channel is either a mixing channel or a completely decohering channel. This confirms the result in Ref. [24].

In the following, we will move on to study the exact form of a commutativity-preserving channel for high-dimension cases.

Definition 2 (isotropic channel). An isotropic channel is of the form

$$\Lambda^{\text{iso}}(\rho) = p\Gamma(\rho) + (1-p)\frac{I}{d}, \quad (12)$$

where Γ is any linear channel that preserves the eigenvalues of ρ . According to Ref. [27], Γ is either a unitary operation or unitarily equivalent to transpose. Parameter p is chosen to make sure that Λ is a completely positive channel. In particular, $-1/(d-1) \leq p \leq 1$ when Γ is a unitary operation, and $-1/(d-1) \leq p \leq 1/(d+1)$ when Γ is unitarily equivalent to transpose.

Theorem 2. Consider the half-classical input state in Eq. (1) with B a qutrit, a channel Λ cannot create a quantum correlation in any half-classical input state if and only if Λ is either a completely decohering channel or an isotropic channel.

Proof. Write the eigendecomposition of $\Lambda(I)$ as

$$\Lambda(I) = \sum_{i=1}^N \lambda_i I_{r_i}. \quad (13)$$

Here $\sum_{i=1}^N r_i = \sum_{i=1}^N r_i \lambda_i = 3$, $\lambda_i \geq 0$, r_i are positive integers, and I_{r_i} are identities of the r_i -dimensional subspace \mathcal{V}_{r_i} . From Eq. (9) we have

$$\Lambda(\rho) = \sum_{i=1}^N q_i \xi_{r_i}^\rho \quad \forall \rho, \quad (14)$$

where $\xi_{r_i}^\rho$ is a density operator on \mathcal{V}_{r_i} .

Clearly, when the eigenvectors of $\Lambda(I)$ are nondegenerate, i.e., $N = 3$ and Eq. (13) becomes $\Lambda(I) = \sum_{i=1}^3 \lambda_i \Pi_i$, the channel Λ is a completely decohering channel, since it takes

any input state ρ to a diagonal form on basis $\{\Pi_i\}$. When two or three eigenvectors of $\Lambda(I)$ are degenerate, we study the eigendecomposition of $\Lambda(\phi)$ for a pure input state $|\phi\rangle$:

$$\Lambda(\phi) = \sum_{i=1}^{N^\phi} \lambda_i^\phi I_{r_i(\phi)}, \quad (15)$$

where $N^\phi \geq N$ and $\mathcal{V}_{r_i(\phi)} \subseteq \mathcal{V}_{r_j}$. When none of $\Lambda(\phi)$ break the degeneracy of eigenvectors of $\Lambda(I)$, i.e., $N^\phi = N$ and $\mathcal{V}_{r_i(\phi)} = \mathcal{V}_{r_i}$, the channel is also a completely decohering channel. Now we focus on the case that some $\Lambda(\phi)$ can break the degeneracy of eigenvectors of $\Lambda(I)$, i.e., $N^\phi > N$ and $\mathcal{V}_{r_i(\phi)} \subset \mathcal{V}_{r_j}$ for some i . Let $\{|\phi_k\rangle\}_{k=0}^2$ be a basis of the three-dimensional Hilbert space and $|\phi_0\rangle$ be the pure input state whose corresponding output state $\Lambda(\phi_0)$ has the most different eigenvalues. It means that $N^{\phi_0} \geq N^\phi, \forall \phi$.

Case 1. For any state $|\phi_0^\perp\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle$ that is orthogonal to $|\phi_0\rangle$, we have $N^{\phi_0^\perp} = N^{\phi_0}$ and $\mathcal{V}_{r_i(\phi_0^\perp)} = \mathcal{V}_{r_i(\phi_0)}$. Then for an arbitrary input state $\varphi = \sum_{i=0}^2 c_i|\phi_i\rangle$, we have

$$[\Lambda(\varphi), \Lambda(c_2^*|\phi_1\rangle - c_1^*|\phi_2\rangle)] = 0. \quad (16)$$

Therefore, $\Lambda(\varphi)$ is diagonal on the same basis as $\Lambda(\phi_0)$.

Case 2. There exists a pure state, e.g., $|\phi_2\rangle$, whose corresponding output state $\Lambda(\phi_2)$ does not break as much degeneracy as $\Lambda(\phi_0)$, i.e., $N^{\phi_2} < N^{\phi_0}$. We will first prove that for any pure state $|\varphi_{01}\rangle = |\phi_0\rangle - \beta_0|\phi_1\rangle$ in two-dimensional subspace $\mathcal{W}_2^{\phi_2}$, the output state is diagonal on the same basis as $\Lambda(\phi_0)$, e.g., $\{\Pi_i\}$. We introduce $|\varphi(\beta)\rangle = |\varphi_{01}\rangle + \beta|\phi_2\rangle$ and $|\varphi_{02}(\beta)\rangle = \beta^*|\phi_0\rangle - |\phi_2\rangle$. Notice that $\langle\varphi_{02}(\beta)|\varphi(\beta)\rangle = 0$, and we have

$$[\Lambda(\varphi_{02}(\beta)), \Lambda(\varphi(\beta))] = 0. \quad (17)$$

Because $\sum_{k=0}^2 |\phi_k\rangle\langle\phi_k| = I$, we have $N^{\phi_1} = N^{\phi_0}$ and $\mathcal{V}_{r_i(\phi_1)} = \mathcal{V}_{r_i(\phi_0)}$. Therefore, $\Lambda(\varphi_{02}(\beta))$ is diagonal on $\{\Pi_i\}$ by noticing that $[\Lambda(\varphi_{02}(\beta)), \Lambda(\phi_1)] = 0$. Since the channel cannot increase the distance between states, $\Lambda(\varphi_{02}(\beta))$ breaks the same degeneracy as $\Lambda(\phi_0)$ for sufficiently large $|\beta|$. From Eq. (17), we have $\Lambda(\varphi(\beta))$ and $\Lambda(\varphi(-\beta))$ are diagonal on $\{\Pi_i\}$. Therefore, $\Lambda(\varphi_{01}) = \Lambda(\varphi(\beta)) + \Lambda(\varphi(-\beta)) - |\beta|^2\Lambda(\phi_2)$ is also diagonal on $\{\Pi_i\}$. Further, we will show that $\Lambda(\varphi(\beta))$ is diagonal on $\{\Pi_i\}$ for arbitrary β . From Eq. (17), this is obvious when $\Lambda(\varphi_{02}(\beta))$ is nondegenerate. For the case where $\Lambda(\varphi_{02}(\beta))$ is degenerate, $\Lambda(\varphi_{02}(-\beta)) = |\beta|^2\Lambda(\phi_0) + \Lambda(\phi_2) - \Lambda(\varphi_{02}(\beta))$ is nondegenerate, and consequently, $\Lambda(\varphi(-\beta))$ is diagonal on $\{\Pi_i\}$. Therefore, $\Lambda(\varphi(\beta)) = \Lambda(\varphi_{01}) + |\beta|^2\Lambda(\phi_2) - \Lambda(\varphi(-\beta))$ is diagonal on $\{\Pi_i\}$. Λ is a completely decohering channel.

Case 3. Now we are left only with the case that $N^{\phi_0^\perp} = N^{\phi_0}$ but $I_{r_i(\phi_0^\perp)} \neq I_{r_i(\phi_0)}$, which can happen only when $\Lambda(I) = I$ and $N^{\phi_k} = 2$. Therefore, we have

$$\Lambda(\phi_k) = p\Pi(\phi_k) + (1-p)\frac{I}{3}, \quad (18)$$

where $\Pi(\phi_k)$ is a basis determined by $|\phi_k\rangle$. Notice that p is independent of $|\phi_k\rangle$ because of the linearity of Λ . Consequently, for any input state $\rho = \sum_i p_i|\alpha_i\rangle\langle\alpha_i|$, we have $\Lambda(\rho) = p\sum_i p_i\Pi(\alpha_i) + (1-p)I/3$. It means that channel Λ is an isotropic channel.

Combining the three cases, we conclude that for a qutrit, a commutativity-preserving channel is either a completely decohering channel or an isotropic channel.

Since the depolarizing channel is a subset of mixing channel, there exist mixing channels that are able to locally create quantum correlation. Therefore, mixedness can contribute to creation of quantum correlations. Here we give an example to look more closely at why a mixing channel can create quantum correlation in states with high dimensions. Consider the mixing channel $\Lambda(\cdot) = \sum_i E^{(i)}(\cdot)E^{(i)\dagger}$, where the Kraus operators are

$$\begin{aligned} E^{(0)} &= |2\rangle\langle 2|, \\ E^{(i)} &= e_i u_2^{(i)}(|0\rangle\langle 0| + |1\rangle\langle 1|) \quad i = 1, 2, \dots \end{aligned} \quad (19)$$

Here $u_2^{(i)}$ are rank-2 unitary operators on basis $\{|0\rangle, |1\rangle\}$. This channel can create quantum correlation in the state $\rho = \tilde{\rho}_A^\phi \otimes |\phi\rangle_B \langle\phi| + \tilde{\rho}_A^\psi \otimes |\psi\rangle_B \langle\psi|$ if and only if Eq. (4) is violated. Writing the two orthogonal states as $|\phi\rangle = \sum_{i=0}^2 a_i|i\rangle$ and $|\psi\rangle = \sum_{i=0}^2 b_i|i\rangle$ ($\sum_{i=0}^2 a_i b_i^* = 0$), we obtain the left-hand side of Eq. (4):

$$\left[\sum_i e_i^2 u_2^{(i)} |\phi_2\rangle\langle\phi_2| u_2^{(i)\dagger}, \sum_i e_i^2 u_2^{(i)} |\psi_2\rangle\langle\psi_2| u_2^{(i)\dagger} \right], \quad (20)$$

where $|\phi_2\rangle = a_0|0\rangle + a_1|1\rangle$ and $|\psi_2\rangle = b_0|0\rangle + b_1|1\rangle$ are reduced states on Hilbert space of dimension 2. Therefore, Eq. (4) is violated if and only if $\langle\phi_2|\psi_2\rangle \neq 0, 1$. Two high-dimension orthogonal states may become unorthogonal when reduced to Hilbert space of dimension two. This is just the reason for creating quantum correlation using a local mixing channel. Isotropic channels act on all the states in Hilbert spaces equivalently, so they are likely the only subset of mixing channels that belongs to the class of commutativity-preserving channels. This observation leads to the following conjecture.

Conjecture. Consider the half-classical input state in Eq. (1) where B is a d -dimensional quantum system (qudit) with $d \geq 3$, then a channel Λ cannot create quantum correlation in any half-classical input state if and only if Λ is either a completely decohering channel or an isotropic channel.

We further prove that a mixing channel cannot increase the teleportation fidelity of any two-qudit state. The average teleportation fidelity f is related to the maximum singlet fraction (MSF) [28] $F = \max_\Phi \langle\Phi|\rho|\Phi\rangle$ as $f = (dF + 1)/(d + 1)$. After the action of a mixing channel on B , the MSF becomes

$$F' = \text{Tr}(\rho \Xi), \quad (21)$$

where $\Xi = \sum_i I \otimes E^{(i)\dagger} |\Phi\rangle\langle\Phi| I \otimes E^{(i)}$. Notice that for a mixing channel $\Lambda(\cdot) = \sum_i E^{(i)}(\cdot)E^{(i)\dagger}$, its conjecture $\Lambda^*(\cdot) = \sum_i E^{(i)\dagger}(\cdot)E^{(i)}$ is also a mixing channel. Therefore, $\Xi_A = \Xi_B = I/2$, so Ξ can be decomposed as a mixture of maximally entangled pure states $\Xi = \sum_i p_i |\Phi_i\rangle\langle\Phi_i|$. Then we have $F' = \sum_i p_i \langle\Phi_i|\rho|\Phi_i\rangle \leq F$. Therefore, average teleportation fidelity can never be increased by a mixing channel. This result suggests that quantum correlation created by a mixing channel may not be a useful resource for quantum-information tasks.

In summary, we have proved that the necessary and sufficient condition for a local operation to create quantum correlation in some half-classical state is that it is not a commutativity-preserving channel. When the subsystem B affected by the

local channel is a qubit, a commutativity-preserving channel is either a mixing channel or a completely decohering channel. This result confirms the results in Ref. [24]. When B is a qutrit, we have proved that a commutativity-preserving channel is either an isotropic channel or a completely decohering channel. This result is likely to be extended to arbitrary finite dimension situation.

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